7.7 Additional Limit Thms

DEF. Convergence in Probability

The sequence of random vars Y_n is said to converge in probability to Y, written $Y_n \xrightarrow{P} Y$ if

$$\lim_{n\to\infty} P\left[|Y_n - Y| < \varepsilon\right] = 1$$

Note: this is a bit more general that stochastic convergence to a constant. Most of the time we talk about when Y is constant.

Note that convergence in prob. is stronger than convergence in distribution.

Thm: If $Y_n \xrightarrow{P} Y$, then $Y_n \xrightarrow{d} Y$.

Compare to last result in section 7.6.

If $Z_n = \frac{\sqrt{n}(Y_n - m)}{c} \xrightarrow{d} N(0,1)$, then $Y_n \xrightarrow{P} m$.

For the special case of $Y = c$,
the limiting dist is the
degenerate dist $P[Y=c]=1$

Thm: If $Y_n \xrightarrow{P} c$, then for any function $g(y)$ that is continuous at c.

$$g(Y_n) \xrightarrow{P} g(c).$$

Proof: Because $g(y)$ is cont. at c, then
$\forall \varepsilon > 0, \exists \delta > 0$, such that

$$|y-c| < \delta \Rightarrow |g(y)-g(c)| < \varepsilon$$

$$\Rightarrow P[\,|g(Y_n)-g(c)| < \varepsilon] \geq P[\,|Y_n - c| < \delta]$$

Because $Y_n \xrightarrow{P} c$, then

$$\lim_{n\to\infty} P[\,|g(Y_n)-g(c)| < \varepsilon] \geq \lim_{n\to\infty} P(|Y_n-c|<\delta) = 1$$

Therefore,

$$\lim_{n\to\infty} P[\,|g(Y_n)-g(c)| < \varepsilon] = 1$$

$$g(Y_n) \xrightarrow{P} g(c)$$

Thm: If X_n & Y_n two seq. R.V.

$X_n \xrightarrow{P} c$

$Y_n \xrightarrow{P} d$.

(1.) $aX_n + bY_n \xrightarrow{P} ac + bd$

(2) $X_nY_n \xrightarrow{P} cd$

(3) $\frac{X_n}{c} \xrightarrow{P} 1$, $c \neq 0$.

(4) $\frac{1}{X_n} \xrightarrow{P} \frac{1}{c}$, $c \neq 0$, $P[X_n = 0] = 0$

(5) $\sqrt{X_n} \xrightarrow{P} \sqrt{c}$, $P[X_n \geq 0] = 1$, $c > 0$

Ex: $Y \sim BIN(n, p)$

$\hat{p} = \frac{Y}{n} \xrightarrow{P} p$ (Bernolli Law of Large #'s)

$\hat{p}(1-\hat{p}) \xrightarrow{P} p(1-p)$

Thm: Slutsky's Thm

X_n, Y_n R.V.

$X_n \xrightarrow{P} c$, $Y_n \xrightarrow{d} Y$,

1. $X_n + Y_n \xrightarrow{d} c + Y$
2. $X_nY_n \xrightarrow{d} cY$
3. $\frac{Y_n}{X_n} \xrightarrow{d} \frac{Y}{c}$, $c \neq 0$.

Ex: $X_i \sim BIN(1,p)$ R.S. of size n.

$$\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} Z \sim N(0,1) \quad \text{(by CLT)}$$

We also know $\hat{p}(1-\hat{p}) \xrightarrow{P} p(1-p)$.

$$\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \xrightarrow{d} Z \sim N(0,1)$$

Thm: If $Y_n \xrightarrow{d} Y$, then for any cont. function $g(y)$,

$$g(Y_n) \xrightarrow{d} g(Y).$$

$g(y)$ is assumed to not depend on n.

Thm: If $\frac{\sqrt{n}(Y_n - m)}{c} \xrightarrow{d} Z \sim N(0,1)$, and if $g'(m) \neq 0$, then

$$\frac{\sqrt{n}\left(g(Y_n) - g(m)\right)}{|c\, g'(m)|} \xrightarrow{d} Z \sim N(0,1)$$

Hence,

$$g(Y_n) \mathrel{\dot\sim} N\left\{ g(m), \frac{c^2[g'(m)]^2}{n} \right\}$$

<u>Ex</u>: By the CLT,

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1)$$

$$\bar{X}_n \mathrel{\dot\sim} N\left\{ \mu, \frac{\sigma^2}{n} \right\}$$

$$g(x) = x^2. \Rightarrow g'(x) = 2x$$

It follows that

$$\bar{X}_n^2 \mathrel{\dot\sim} N\left\{ \mu^2, \frac{\sigma^2(2\mu)^2}{n} = \frac{4\sigma^2\mu^2}{n} \right\}$$