

[7.5] Asymptotic Normal Dist.

DEF: If Y_1, Y_2, \dots is a sequence of random variables and m, c are constants such that

$$Z_n = \frac{Y_n - m}{c/\sqrt{n}} \xrightarrow{d} Z \sim N(0, 1)$$

as $n \rightarrow \infty$, then Y_n is said to have an asymptotic normal dist with asymptotic mean m , and asymptotic variance c^2/n

EX: Ex 4.6.3.

$n=40$ lifetimes of electrical parts
 $X_i \sim \text{EXP}(100)$

By CLT,

\bar{X}_n has an asymptotic normal dist with mean $m=100$, and variance $\frac{c^2}{n} = \frac{100^2}{40} = 250$

Thm 7.51.

X_1, \dots, X_n RS from cont. dist.

x_p is the p^{th} percentile

If $\frac{k}{n} \rightarrow p$, then the sequence of k^{th} order stats $X_{k:n}$ is asymptotically normal with mean x_p and variance c^2/n , where

$$c^2 = \frac{p(1-p)}{[f(x_p)]^2}$$
$$\frac{X_{k:n} - x_p}{c/\sqrt{n}} \xrightarrow{d} N(0,1)$$

7.6 Properties of Stochastic Convergence

Sequence of r.v. that converge stochastically to a constant.

Thm: The sequence Y_1, Y_2, \dots converges stochastically to c iff $\forall \epsilon > 0, \lim_{n \rightarrow \infty} P[|Y_n - c| < \epsilon] = 1.$

If a sequence of R.V. satisfies \rightarrow then the sequence is said to converge in probability to c .
In statistical notation,

$$Y_n \xrightarrow{P} c.$$

Ex. Bernoulli Law of Large Numbers.

Use Chebyshev's Inequality.

$$P[|X-\mu| < \varepsilon] \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

Let X_1, \dots, X_n RS from BERNOLLI(p)

$$\hat{p}_n = \frac{1}{n} \sum X_i$$

$$E(\hat{p}_n) = \frac{1}{n} \sum E(X_i) = \frac{np}{n} = p$$

$$V(\hat{p}_n) = \frac{1}{n^2} \sum V(X_i) = \frac{npq}{n^2} = \frac{pq}{n}$$

It follows that

$$P[|\hat{p}_n - p| < \varepsilon] \geq 1 - \frac{pq/n}{\varepsilon^2} = 1 - \frac{pq}{n\varepsilon^2}$$

$$\lim_{n \rightarrow \infty} P[|\hat{p}_n - p| < \varepsilon] \geq \lim_{n \rightarrow \infty} \left(1 - \frac{pq}{n\varepsilon^2}\right) = 1.$$

$$\text{Thus } \lim_{n \rightarrow \infty} P[|\hat{p}_n - p| < \varepsilon] = 1.$$

Therefore, $\hat{p}_n \xrightarrow{P} p$

Thm: Law of Large Numbers.

If X_1, X_2, \dots, X_n is a R.S from
from a dist with finite mean μ ,
and variance, then the sequence of
sample means converge in prob to μ .

$$\bar{X}_n \xrightarrow{P} \mu$$

Proof:

$$E(\bar{X}) = E\left[\frac{1}{n} \sum X_i\right] = \frac{1}{n} \sum E(X_i) = \frac{n\mu}{n} = \mu$$

$$V(\bar{X}) = V\left(\frac{1}{n} \sum X_i\right) = \frac{1}{n^2} \sum V(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$P\left[|\bar{X} - \mu| < \varepsilon\right] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

$$\lim_{n \rightarrow \infty} P\left[|\bar{X} - \mu| < \varepsilon\right] \geq \lim_{n \rightarrow \infty} \left(1 - \frac{\sigma^2}{n\varepsilon^2}\right) = 1$$

Thus,

$$\bar{X} \xrightarrow{P} \mu.$$

$$\text{If } Z_n = \frac{\sqrt{n}(\bar{Y}_n - m)}{c} \xrightarrow{d} Z \sim N(0,1),$$

$$\text{then } \bar{Y}_n \xrightarrow{P} m.$$

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$$\text{Thus } \lim_{n \rightarrow \infty} P[|\hat{p}_n - p| < \varepsilon] = 1.$$

Therefore, $\hat{p}_n \xrightarrow{P} p$