

7.3 Central Limit Thm

Thm 7.3.1 Let Y_1, Y_2, \dots be a sequence of R.V.

with CDFs

$$G_1(y), G_2(y), \dots \quad \text{and}$$

with MGFs

$$M_1(t), M_2(t), \dots$$

If $M(t)$ is the mgf of a CDF $G(y)$, and

$$\text{If } \lim_{n \rightarrow \infty} M_n(t) = M(t) \text{ for all } t \in (-h, h)$$

$$\text{then } \lim_{n \rightarrow \infty} G_n(y) = G(y) \text{ for all points of continuity of } G(y).$$

Preliminary:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{c}{n}\right)^{nb} = e^{cb}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{c}{n} + \frac{d(n)}{n}\right)^{nb} = e^{cb}$$

if $d(n) \rightarrow 0$ as $n \rightarrow \infty$

Ex: Let X_1, X_2, \dots, X_n be a RS from $X_i \sim \text{BIN}(1, p)$

$$\text{Consider } Y_n = \sum_{i=1}^n X_i \sim \text{BIN}(n, p)$$

Let $p \rightarrow 0$ as $n \rightarrow \infty$ such that $\mu = np$ remains fixed $\mu > 0$.

$$\begin{aligned} \text{Then } M_n(t) &= (pe^{t+q})^n = \left(\frac{\mu e^t}{n} + 1 - \frac{\mu}{n}\right)^n \\ &= \left[1 + \frac{\mu(e^t - 1)}{n}\right]^n = e^{\mu(e^t - 1)} \end{aligned}$$

$$Y_n \xrightarrow{d} Y \sim \text{POI}(\mu)$$

Thm 7.3.2 Central Limit Thm (CLT)

If X_1, X_2, \dots, X_n is a RS from a dist with mean μ and variance $\sigma^2 < \infty$, then the limiting dist of

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum X_i - n\mu}{\sqrt{n}\sigma} \quad \bar{X} = \frac{1}{n} \sum X_i$$

is the standard normal, $Z_n \xrightarrow{d} Z \sim N(0,1)$ as $n \rightarrow \infty$.

Proof: The result holds for any dist with finite mean & variance, but the proof is easier by assuming the MGF exists! (Use the characteristic function otherwise (Fourier Transform))

Let $m(t)$ denote the MGF of $X - \mu$.

Then

$$m(t) = M_{X-\mu}(t)$$

$$m(0) = 1$$

$$m'(0) = E(X - \mu) = E(X) - \mu = \mu - \mu = 0$$

$$m''(0) = E(X - \mu)^2 = \sigma^2$$

So, expanding $m(t)$ in a Taylor Series about 0 gives.

$$m(t) = m(0) + m'(0)t + \frac{m''(\xi)}{2} t^2$$

$\xi \in (0, t)$

Hence, $m(t) = 1 + \frac{m''(\xi) t^2}{2}$

Now, the R.V. Z_n can be written as.

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^n (X_i - \mu)}{\sqrt{n} \sigma}$$

So the MGF of Z_n is

$$M_{Z_n}(t) = E \left[\exp \left\{ \frac{\sum (X_i - \mu)}{\sigma \sqrt{n}} t \right\} \right]$$

$$= M_{\sum (X_i - \mu)} \left(\frac{t}{\sigma \sqrt{n}} \right)$$

$$= \prod_{i=1}^n M_{X_i - \mu} \left(\frac{t}{\sigma \sqrt{n}} \right)$$

$$= \left[M_{X - \mu} \left(\frac{t}{\sigma \sqrt{n}} \right) \right]^n$$

$$= \left[m \left(\frac{t}{\sigma \sqrt{n}} \right) \right]^n$$

$$= \left[1 + \frac{m''(\xi) t^2}{2 n \sigma^2} \right]^n, \quad 0 < |\xi| < \frac{|t|}{\sigma \sqrt{n}}$$

Now, since $\xi \in \left(\frac{-t}{\sigma \sqrt{n}}, \frac{t}{\sigma \sqrt{n}} \right)$, the $\xi \rightarrow 0$ as $n \rightarrow \infty$, hence, $\lim_{n \rightarrow \infty} m''(\xi) = m''(0) = \sigma^2$

It follows that

$$= \left[1 + \frac{\sigma^2 t^2}{2 n \sigma^2} - \frac{\sigma^2 t^2}{2 n \sigma^2} + \frac{m''(\xi) t^2}{2 n \sigma^2} \right]^n$$

$$= \left[1 + \frac{\cancel{\sigma^2} t^2}{2 n \cancel{\sigma^2}} + \underbrace{\frac{(m''(\xi) - \sigma^2) t^2}{2 n \sigma^2}}_{\frac{d(n)}{n} \rightarrow 0} \right]^n = e^{\frac{t^2}{2}}$$

$$Z_n \xrightarrow{d} Z \sim N(0, 1)$$

ex: Roll 20 dice (6 sided die)

$$X_i \sim DU(6) \rightarrow E(X_i) = \frac{6+1}{2} = \frac{7}{2}$$

$$\text{Find } P[\sum X_i > 85] \quad V(X_i) = \frac{6^2-1}{12} = \frac{35}{12}$$

$$Z_n = \frac{\sum X_i - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} Z \sim N(0,1)$$

$$P[\sum X_i > 85] \approx P\left[\underbrace{\frac{\sum X_i - 20(\frac{7}{2})}{\sqrt{20} \sqrt{\frac{35}{12}}}}_Z > \underbrace{\frac{85 - 20(\frac{7}{2})}{\sqrt{20} \sqrt{\frac{35}{12}}}} \right]$$
$$= P[Z > 1.96] = .0250$$