6.3 Joint transformations.

Thm 6.3.5

If X is a vector of discrete R.V. with joint pdf $f_X(x)$ and $Y = u(X)$ defines a one to one transformation, then joint pdf of Y is

$$f_Y(y) = f_X(x), \text{ when}$$

$$y = (y_1, y_2, \cdots, y_k), x = (x_1, x_2, \cdots, x_k)$$

and x is the solution of the transformation

$$y = u(x). \quad (x \text{ depends on } y.)$$

If it is not 1-1, split it up over intervals where it is. A_j

Then the equation $y = u(x)$ has a unique sol'n $x = x_j$ or $x_j = (x_{1j}, x_{2j}, \dots x_{kj})$ over A_j. Then the pdf is

$$f_Y(y) = \sum_j f_X(x_j)$$

joint transforms of continuous R.V.'s can be accomplished, but the Jacobian has to be generalized. Suppose $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ are functions and x_1 & x_2 are unique solutions to $y_1 = u_1(x_1, x_2)$
$y_2 = u_2(x_1, x_2)$.

Then the Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Ex: Suppose we want to transform x_1 & x_2 into x_1 & $x_1 x_2$. Then

$$y_1 = x_1 \implies x_1 = y_1$$

$$y_2 = x_1 x_2 \implies x_2 = \frac{y_2}{x_1} = \frac{y_2}{y_1}$$

Note:

$$Y = U(\underbrace{x_1, x_2}_{X}), \text{ where } U(x_1, x_2) = (x_1, x_1 x_2)$$

$$X = U^{-1}(Y) = U^{-1}(Y_1, Y_2) = (Y_1, Y_2/Y_1)$$

So

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1} \end{vmatrix} = \frac{1}{y_1}$$

For a general transform $y = u(x)$, that has a unique solution

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \cdots & \frac{\partial x_1}{\partial y_k} \\ \vdots & \frac{\partial x_2}{\partial y_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial x_k}{\partial y_1} & \cdots & \cdots & \frac{\partial x_k}{\partial y_k} \end{vmatrix}$$

EX: $X_1 \sim EXP(1)$ $X_1 \perp\!\!\!\perp X_2$

$X_2 \sim EXP(1)$

$$f(x_1, x_2) = e^{-(x_1+x_2)}$$

Transform from (x_1, x_2) to $(x_1 - x_2, x_1 + x_2)$

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} = u(x) = u\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} y_1 &= x_1 - x_2 \\ y_2 &= x_1 + x_2 \\ \hline y_1 + y_2 &= 2x_1 \end{aligned}$$

$$x_1 = \frac{y_1 + y_2}{2}$$

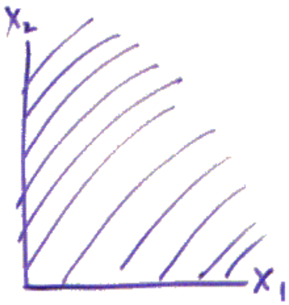
$$\begin{aligned} -y_1 &= -x_1 + x_2 \\ y_2 &= x_1 + x_2 \\ \hline y_2 - y_1 &= 2x_2 \end{aligned}$$

$$x_2 = \frac{y_2 - y_1}{2}$$

So $x = u^{-1}(Y) = \begin{pmatrix} \frac{Y_1 + Y_2}{2} \\ \frac{Y_2 - Y_1}{2} \end{pmatrix}$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\begin{aligned} f_Y(y) &= f_X(x)\,|J| \\ &= f_X\left(\frac{y_1+y_2}{2}, \frac{y_2-y_1}{2}\right) \cdot \frac{1}{2} \\ &= e^{-\left(\frac{y_1+y_2}{2} + \frac{y_2-y_1}{2}\right)} \cdot \frac{1}{2} \\ &= \frac{1}{2} e^{-y_2}, \quad y \in B \Rightarrow \text{what is } B? \end{aligned}$$



$$A = \{(x_1, x_2) \mid f(\underset{\uparrow}{x}) > 0\}$$

x_1, x_2

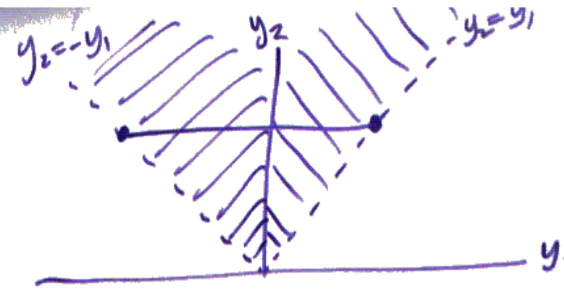
$x_1 > 0, \; x_2 > 0.$

The transform $x_1 = \dfrac{y_1 + y_2}{2} > 0$

$$2x_1 = y_1 + y_2 > 0 \Rightarrow y_1 > -y_2$$

$$x_2 = \frac{y_2 - y_1}{2} > 0$$

$$\Rightarrow 2x_2 = y_2 - y_1 > 0 \Rightarrow y_2 > y_1$$



So $y_2 > 0$ and $-y_2 < y_1 < y_2$

$|y_1| < y_2 < \infty$

$$f_Y(y) = \frac{1}{2} e^{-y_2} I_{(0,\infty)}(y_2) \, I_{(|y_1|,\infty)}(y_2)$$

The marginal for Y_1 is:

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} e^{-y_2} I_{(|y_1|,\infty)}(y_2)\, I_{(0,\infty)}(y_2)\, dy_2$$

$$= \int_{|y_1|}^{\infty} \frac{1}{2} e^{-y_2}\, dy_2 = -\frac{1}{2} e^{-y_2} \Big|_{|y_1|}^{\infty}$$

$$= \frac{1}{2} e^{-|y_1|}$$

$$Y_1 \sim DE(1, 0)$$

The marginal for Y_2 is

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_Y(y)\, dy_1$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} e^{-y_2} \underbrace{I_{(|y_1|,\infty)}(y_2)}_{\downarrow}\, I_{(0,\infty)}(y_2)\, dy_1$$

$$|y_1| < y_2 \Rightarrow \underbrace{-y_2 < y_1 < y_2}_{I_{(-y_2, y_2)}(y_1)}$$

$$= \int_{-y_2}^{y_2} \frac{1}{2} e^{-y_2} I_{(0,\infty)}(y_2)\, dy_1$$

$$= \frac{1}{2} e^{-y_2} I_{(0,\infty)}(y_2) \int_{-y_2}^{y_2} dy_1$$

$$= \frac{1}{8} e^{-y_2} I_{(0,\infty)}(y_2)\; 8\, y_2$$

$$= y_2 e^{-y_2} I_{(0,\infty)}(y_2) \Longrightarrow Y_2 \sim GAM(1, 2)$$

Ex: Let $X_i \sim GAM(\theta, \kappa_i)$ $i = 1, 2$

$X_1 \perp\!\!\!\perp X_2$

We want to find the dist of

$$\frac{X_1}{X_1+X_2} \quad \text{and} \quad \underbrace{\frac{X_2}{X_1+X_2}}_{\text{do later.}}$$

$$Z_1 = \frac{X_1}{X_1+X_2} \qquad Z_2 = X_2$$

$$X_1 = \frac{Z_1 Z_2}{1-Z_1} \qquad X_2 = Z_2$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{vmatrix} = \begin{vmatrix} \frac{z_2}{(1-z_1)^2} & \frac{z_1}{1-z_1} \\ 0 & 1 \end{vmatrix} = \frac{z_2}{(1-z_1)^2}$$

The pdf of X_i is

$$f_{X_i}(x_i) = \frac{1}{\Gamma(\kappa_i)\theta^{\kappa_i}} x_i^{\kappa_i - 1} e^{-\frac{x_i}{\theta}} I_{(0,\infty)}(x_i)$$

The joint pdf of X_1 & X_2 is

$$f_{X_1,X_2}(x_1,x_2) = f_1(x_1) \cdot f_2(x_2)$$

$$= \frac{1}{\Gamma(\kappa_1)\Gamma(\kappa_2)\theta^{\kappa_1+\kappa_2}} x_1^{\kappa_1-1} x_2^{\kappa_2-1} e^{-\frac{x_1+x_2}{\theta}} I_{(0,\infty)}(x_1) \times I_{(0,\infty)}(x_2)$$

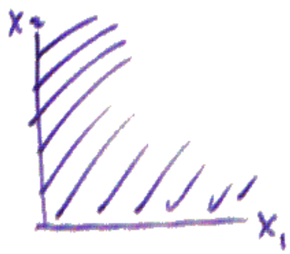
So

$$f_z(z) = f_X(x)\,|J|$$

$$= f_X\left(\frac{z_1 z_2}{1-z_1},\, z_2\right)\frac{z_2}{(1-z_1)^2}$$

$$= \frac{1}{\Gamma(k_1)\Gamma(k_2)\,\theta^{k_1+k_2}}\left(\frac{z_1 z_2}{1-z_1}\right)^{k_1-1} z_2^{k_2-1}\, e^{-\frac{1}{\theta}\left(\frac{z_1 z_2}{1-z_1}+z_2\right)}\frac{z_2}{(1-z_1)^2}\, I_{(0,\infty)}\left(\frac{z_1 z_2}{1-z_1}\right) I_{(0,\infty)}(z_2)$$

$$= \frac{1}{\Gamma(k_1)\Gamma(k_2)\,\theta^{k_1+k_2}}\,\frac{z_1^{k_1-1}\, z_2^{k_1+k_2-1}}{(1-z_1)^{k_1+1}}\, e^{-\frac{1}{\theta}\underbrace{\left(\frac{\cancel{z_1 z_2}+z_2(1-\cancel{z_1})}{1-z_1}\right)}_{\frac{z_2}{1-z_1}}}\, \underbrace{I_{(0,\infty)}\left(\frac{z_1 z_2}{1-z_1}\right)}_{I_{(0,1)}(z_1)} I_{(0,\infty)}(z_2)$$



Note that $\underbrace{0 < \frac{z_1 z_2}{1-z_1} < \infty}$

$0 < z_1 < 1 \qquad 0 < z_2 < \infty$

$\Rightarrow$

We want the dist of z_1

$$f(z_1) = \int_{-\infty}^{\infty} f_z(z)\, dz_2$$

$$= \underbrace{\frac{I_{(0,1)}(z_1)\; z_1^{k_1-1}}{\Gamma(k_1)\Gamma(k_2)\theta^{k_1+k_2}(1-z_1)^{k_1+1}}}_{c(z_1)} \int_0^{\infty} z_2^{k_1+k_2-1} e^{-\frac{z_2}{\theta(1-z_1)}}\, dz_2$$

$$u = \frac{z_2}{\theta(1-z_1)} \Rightarrow z_2 = u\theta(1-z_1)$$

$$dz_2 = \theta(1-z_1)\,du$$

$$B(k_1,k_2) = \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)}$$

$$= \int_0^1 z_1^{k_1-1}(1-z_1)^{k_2-1}\,dz_1$$

Try and prove it! If you do, I'll add 5% to any one of your tests.

$$= c(z_1) \int_0^{\infty} \left[u\,\theta(1-z_1)\right]^{k_1+k_2-1} e^{-u}\, \theta(1-z_1)\, du$$

$$= c(z_1)\, \theta^{k_1+k_2}(1-z_1)^{k_1+k_2} \underbrace{\int_0^{\infty} u^{k_1+k_2-1} e^{-u}\, du}_{\Gamma(k_1+k_2)} = \frac{I_{(0,1)}(z_1)\, z_1^{k_1-1}\,\Gamma(k_1+k_2)\cancel{\theta^{k_1+k_2}}(1-z_1)^{\cancel{k_1}+k_2}}{\Gamma(k_1)\Gamma(k_2)\cancel{\theta^{k_1+k_2}}(1-z_1)^{\cancel{k_1}+1}}$$

$$= \frac{\Gamma(k_1+k_2)}{\Gamma(k_1)\Gamma(k_2)} z_1^{k_1-1}(1-z_1)^{k_2-1} I_{(0,1)}(z_1)$$

$$z_1 \sim BETA(k_1,k_2)$$