Chap 6

Functions of Random Vars.

Suppose X represents the age in weeks of some component. Perhaps age in days is desired instead.

$$W = \ln X$$

$$\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n$$

We will discuss methods of deriving the pdf of the new random variable that is a function of the other random vars.

6.2 The CDF Technique.

Assume X has CDF $F_X(x)$ and we want to find the CDF of $Y = u(X)$

Idea: Express the CDF of Y in terms of the dist of X.

Define $A_y = \{x \mid u(x) \le y\}$. It follows that $[Y \geq y]$ and $[X \in A_y]$ are equivalent sets and as such

$$F_Y(y) = P[u(X) \le y] = P[X \in A_y]$$

often $X \in A_y$ can be expressed as $x_1 \le x \le x_2$ where the limits depend on y

In the cont. case, if

$u(X) \le y$ is equiv. to $x_1 \le X \le x_2$

$$F_Y(y) = P[u(X) \le y] = P[x_1 \le X \le x_2]$$

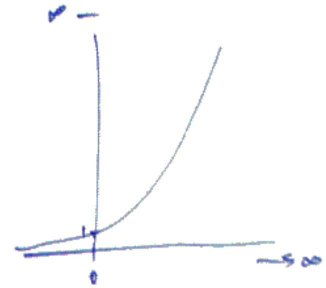
$$= \int_{x_1}^{x_2} f_X(x)\,dx = F_X(x_2) - F_X(x_1)$$

and the pdf is

$$\frac{d F_Y(y)}{dy} = f_Y(y)$$

EX: Suppose that the CDF is

$$F(x) = 1 - e^{-2x}, \quad 0 < x < \infty$$



Consider $Y = e^X$. The CDF of Y is therefore

$$F_Y(y) = P[Y \le y] = P[e^X \le y]$$

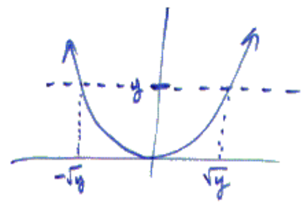
$$= P[X \le \ln y] = F_X(\ln y)$$

$$= 1 - e^{-2\ln y} = 1 - e^{\ln y^{-2}}$$

$$= 1 - y^{-2}$$

Thus, the pdf of Y is $f_Y(y) = \dfrac{2}{y^3}, \quad 1 < y < \infty$

Ex: Consider a cont. R.V. X.



Let $Y = X^2$. then

$$F_Y(y) = P\left[X^2 \le y\right] = P\left(|X| \le \sqrt{y}\right]$$

$$= P\left[-\sqrt{y} \le X \le \sqrt{y}\right] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

The pdf of Y is

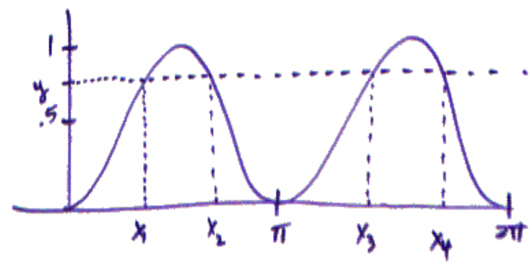
$$f(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\sqrt{y}) - \frac{d}{dy} F_X(-\sqrt{y})$$

$$= f(\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right) - f_X(-\sqrt{y})\left(\frac{1}{-2\sqrt{y}}\right)$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right], \quad y \ge 0$$

Ex: Suppose $X \sim UNIF(0, 2\pi)$

Consider $Y = \sin^2 x$



$$F_Y(y) = P\left[Y \le y\right] = P(X \le x_1) + P(x_2 \le X \le x_3) + P(X \ge x_4)$$

same

$$= P\left[X \le x_1\right) + P\left(x_2 \le X \le \pi\right] + P\left[\pi \le X \le x_3\right] + P(X \ge x_4)$$

same

$$= 2P(X \le x_1) + 2P(x_2 \le X \le \pi)$$

x_1 & x_2 are the the solutions to $y = \sin$

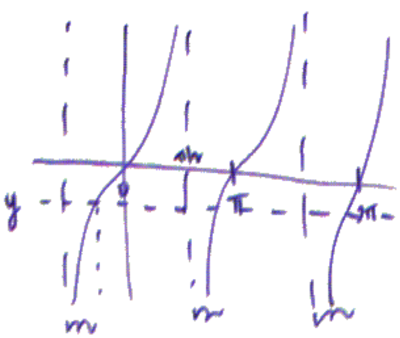
$0 \le x$

Ex. 6.2.3.

$\theta \sim UNIF(0, 2\pi)$

$f(\theta) = \frac{1}{2\pi}, \quad 0 \le \theta \le 2\pi \longrightarrow F_\theta(\theta) = \int_0^\theta \frac{1}{2\pi}\, d\theta = \frac{\theta}{2\pi}$

$Y = \tan\theta.$



$$F_Y(y) = P[\tan\theta \le y]$$

$$= P\left[\frac{\pi}{2} \le \theta < \pi + \tan^{-1} y\right] + P\left[\frac{3\pi}{2} \le \theta \le 2\pi + \tan^{-1} y\right]$$

$$= F_\theta(\pi + \tan^{-1} y) - F_\theta\left(\frac{\pi}{2}\right) + F_\theta(2\pi + \tan^{-1} y) - F_\theta\left(\frac{3\pi}{2}\right)$$

$$= \frac{1}{2\pi}\left[\pi + \tan^{-1} y - \frac{\pi}{2}\right] + \frac{1}{2\pi}\left[2\pi + \tan^{-1} y - \frac{3\pi}{2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{2} + \tan^{-1} y\right] + \frac{1}{2\pi}\left[\frac{\pi}{2} + \tan^{-1} y\right] = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1} y\right)$$

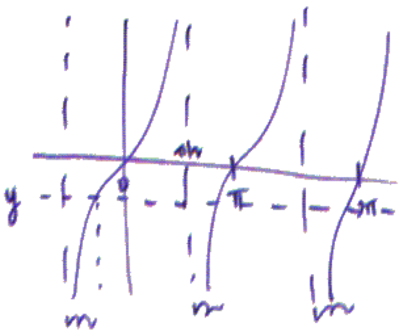
the pdf is $f_Y(y) = \frac{d}{dy}\left(\frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1} y\right)\right) = \frac{1}{\pi}\frac{1}{1+y^2} \sim CAUCHY(0,1)$

Ex: 6.2.3.

$\theta \sim UNIF(0,2\pi)$

$f(\theta)=\frac{1}{2\pi}, \quad 0\le\theta\le 2\pi \quad \rightarrow \quad F_\theta(\theta)=\int_0^\theta \frac{1}{2\pi}\,d\theta = \frac{\theta}{2\pi}$

$Y=\tan\theta$.



$$F_Y(y) = P[\tan\theta \le y]$$
$$= P\left[\frac{\pi}{2}\le\theta<\pi+\tan^{-1}y\right] + P\left[\frac{3\pi}{2}\le\theta\le 2\pi+\tan^{-1}y\right]$$
$$= F_\theta(\pi+\tan^{-1}y)-F_\theta\left(\frac{\pi}{2}\right) + F_\theta(2\pi+\tan^{-1}y)-F_\theta\left(\frac{3\pi}{2}\right)$$
$$= \frac{1}{2\pi}\left[\pi+\tan^{-1}y-\frac{\pi}{2}\right] + \frac{1}{2\pi}\left[2\pi+\tan^{-1}y-\frac{3\pi}{2}\right]$$
$$= \frac{1}{2\pi}\left[\frac{\pi}{2}+\tan^{-1}y\right] + \frac{1}{2\pi}\left[\frac{\pi}{2}+\tan^{-1}y\right] = \frac{1}{\pi}\left(\frac{\pi}{2}+\tan^{-1}y\right)$$

the pdf is $f_Y(y)=\frac{d}{dy}\left(\frac{1}{\pi}\left(\frac{\pi}{2}+\tan^{-1}y\right)\right) = \frac{1}{\pi}\frac{1}{1+y^2} \sim CAUCHY(1,0)$

$\boxed{\text{Thm 6.2.1}}$

Let $X=(X_1,\dots,X_k)$ be a k-dimensional R.V. with joint pdf $f(x_1,x_2,\dots,x_k)$.

If $Y=u(X)$ is a function of X, then

$$F_Y(y)=P[u(X)\le y]=\int\cdots\int_{A_y} f(x_1,\dots,x_k)\,dx_1\cdots dx_k$$

where $A_y=\{x \mid u(x)\le y\}$

boldface

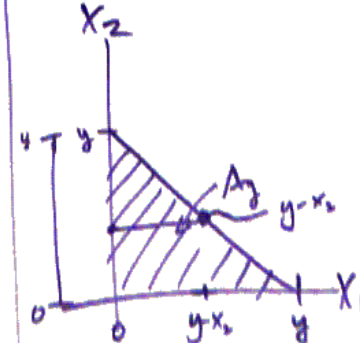
Ex: $X_1 \sim EXP(1)$

$X_2 \sim EXP$

$X_1 \perp\!\!\!\perp X_2$

Find the dist of $Y=X_1+X_2$

$$A_y=\{(x_1,x_2)\mid \underbrace{0\le x_1+x_2\le y}_{\substack{0\le x_1\le y-x_2 \text{ and} \\ 0\le x_2\le y}}\}$$



$$F_Y(y)=\int_0^y\int_0^{y-x_2} e^{-(x_1+x_2)}\,dx_1\,dx_2$$

$$=1-e^{-y}-ye^{-y}$$

$f_Y(y)=ye^{-y},\ y>0$

$Y\sim GAM(1,2)$