

Thm 5.3.1

If ρ is the correlation coeff of X & Y ,

then $-1 \leq \rho \leq 1$

and $\rho = \pm 1$ iff $Y = aX + b$ with prob. 1.

proof:

$$\begin{aligned} h(t) &= E \left[(X - \mu_X)t + (Y - \mu_Y) \right]^2 \\ &= E \left[(X - \mu_X)^2 t^2 + 2t(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2 \right] \\ &= t^2 E(X - \mu_X)^2 + 2tE \left[(X - \mu_X)(Y - \mu_Y) \right] + E(Y - \mu_Y)^2 \end{aligned}$$

Since $h(t) \geq 0$ for all values of t (why?)
 $h(t)$ is a quadratic function, $h(t)$ has at most one real root and thus must have a

non positive discriminant.

$$b^2 - 4ac \leq 0$$
$$\left[2 \underbrace{E \left[(X - \mu_X)(Y - \mu_Y) \right]}_{\sigma_{XY}} \right]^2 - 4 \underbrace{E(X - \mu_X)^2}_{\sigma_X^2} \underbrace{E(Y - \mu_Y)^2}_{\sigma_Y^2} \leq 0$$

$$4\sigma_{XY}^2 - 4\sigma_X^2\sigma_Y^2 \leq 0$$

$$4\sigma_{XY}^2 \leq 4\sigma_X^2\sigma_Y^2$$

$$|\sigma_{XY}| \leq \sigma_X\sigma_Y$$

$$-\sigma_X\sigma_Y \leq \sigma_{XY} \leq \sigma_X\sigma_Y$$

$$-1 \leq \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \leq 1$$

$$-1 \leq \rho \leq 1$$

5.4 Conditional Expectation.

Def: Conditional exp.

$$E(Y|X) = \sum_{\text{all } y} y f(y|x) \quad (X \& Y \text{ are discrete})$$

$$= \int_{-\infty}^{\infty} y f(y|x) \quad (X \& Y \text{ are cont.})$$

Other Notations

$$E[Y|X=x]$$

$$E_{Y|X}(Y)$$

Note $E[Y|X]$ is a function of x !

$$g(x) = E[Y|X]$$

Thm 5.4.1 If X & Y are jointly dist R.V.'s

then $E_x[E(Y|X)] = E(Y)$

$g(x) = E(Y|X) =$ function of X .

$$E_x(g(x)) = \sum_{-\infty}^{\infty} g(x) f(x) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$\begin{aligned} E(E(Y|X)) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(y|x) f(x) dx dy \\ &= \int_{-\infty}^{\infty} y f(y) dy \end{aligned}$$

proof:

$$E[E(Y|X)] = \int_{-\infty}^{\infty} E(Y|x) f_1(x) dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y f(y|x) dy \right] f_1(x) dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(y|x) f_1(x) dy dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y) dy dx$$

$$= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y) dx dy$$

$$= \int_{-\infty}^{\infty} y f_2(y) dy$$

$$= E(Y)$$

Ex: 5.4.2

$X \sim \text{POI}(20)$

$E(X) = \mu = 20$

$Y|X \sim \text{BIN}(X, .85)$

$E(Y) \Rightarrow 2$ ways to find this.

1) find pdf of Y and then compute the mean

2) $E(E(Y|X)) = E(Y)$

$$E(Y) = E(E(Y|X)) = E(X(.85)) = .85 E(X)$$

$$= 20(.85) = 17$$

Thm 5.4.2 If X & Y are independent,

$$E(Y|X) = E(Y)$$

$$E(X|Y) = E(X)$$

Def. The conditional variance of Y given $X=x$ is

$$\begin{aligned} V(Y|X) &= E\{[Y - E(Y|X)]^2 | X\} \\ &= E[Y^2 | X] - [E(Y|X)]^2 \end{aligned}$$

Thm 5.4.3 If X & Y are jointly dist,
then

$$\text{Var}(Y) = E_X[\text{Var}(Y|X)] + \text{Var}_X[E(Y|X)]$$

Ex:

$$\text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X))$$

$$= E(X(.85)(.15)) + \text{Var}(X(.85))$$

$$= .85(.15) E(X) + (.85)^2 \text{Var}(X)$$

$$= .85(.15)(20) + (.85)^2(20)$$

$$= 2.55 + 14.45$$

$$= 17$$