

5.2 Properties of Expected Values.

Thm. If $X = (X_1, \dots, X_k)$ has joint pdf $f(x_1, x_2, \dots, x_k)$ and if $Y = u(x_1, \dots, x_k)$ is a function of X , then

$$E(Y) = E_X[u(x_1, \dots, x_k)], \text{ where}$$

$$E_X[u(x_1, \dots, x_k)] = \sum_{x_1} \dots \sum_{x_k} u(x_1, \dots, x_k) f(x_1, \dots, x_k), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} u(x_1, \dots, x_k) f(x_1, \dots, x_k) dx_1 dx_2 \dots dx_k, \text{ if } X \text{ is contin.}$$

Thm 5.2.2

If X_1, X_2 are R.V with joint pdf $f(x_1, x_2)$,

then

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

Proof: (Discrete case)

$$E(X_1 + X_2) = \sum_{x_1} \sum_{x_2} (x_1 + x_2) f(x_1, x_2)$$

$$= \sum_{x_1} \sum_{x_2} x_1 f(x_1, x_2) + \sum_{x_1} \sum_{x_2} x_2 f(x_1, x_2)$$

$$= \sum_{x_1} x_1 \sum_{x_2} f(x_1, x_2)$$

$$+ \sum_{x_2} x_2 \sum_{x_1} f(x_1, x_2)$$

$$= \sum_{x_1} x_1 f_1(x_1) + \sum_{x_2} x_2 f_2(x_2)$$

$$= E(X_1) + E(X_2)$$

Thm If X & Y are indep R.V.
and $g(x)$ & $h(y)$ are functions, then

$$E[g(x)h(y)] = E(g(x))E(h(y))$$

proof - cont. case is in the book.

DEF. The covariance of a pair of random variables X & Y is defined by

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \sigma_{XY}$$

Thm If X & Y are RV, and a, b are const

then

a) $\text{Cov}(aX, bY) = ab \text{Cov}(X, Y)$

b) $\text{Cov}(X+a, Y+b) = \text{Cov}(X, Y)$

c) $\text{Cov}(X, aX+b) = a \text{Var}(X)$

Thm 5.2.5 If X & Y are RV's, then

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) \text{ and}$$

If X & Y are independent, then

$$\text{Cov}(X, Y) = 0.$$

Proof: $\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$
 $= E[XY - X\mu_Y - \mu_X Y + \mu_X \mu_Y]$
 $= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X \mu_Y$
 $= E(XY) - \mu_X \mu_Y - \cancel{\mu_X \mu_Y} + \cancel{\mu_X \mu_Y}$
 $= E(XY) - \mu_X \mu_Y$

If X, Y are indep, then $\text{Cov}(X, Y) = E(X)E(Y) - \mu_X \mu_Y$
 $= \mu_X \mu_Y - \mu_X \mu_Y = 0$

Thm 5.26

If X_1, X_2 are RV w/pdf $f(x_1, x_2)$,

$$V(X_1 + X_2) = V(X_1) + V(X_2) + 2\text{Cov}(X_1, X_2)$$

If $X_1 \perp\!\!\!\perp X_2$ ($\perp\!\!\!\perp$ = is independent of), then

$$V(X_1 + X_2) = V(X_1) + V(X_2)$$

proof: DEF: $\mu_1 = E(X_1)$, $\mu_2 = E(X_2)$

$$\begin{aligned} V(X_1 + X_2) &= E(X_1 + X_2)^2 - [E(X_1 + X_2)]^2 \\ &= E(X_1^2 + 2X_1X_2 + X_2^2) - \left[\frac{E(X_1)}{\mu_1} + \frac{E(X_2)}{\mu_2} \right]^2 \end{aligned}$$

$$= E(X_1^2) + 2E(X_1X_2) + E(X_2^2) - \mu_1^2 - 2\mu_1\mu_2 - \mu_2^2$$

$$= [E(X_1^2) - \mu_1^2] + [E(X_2^2) - \mu_2^2] + 2[E(X_1X_2) - \mu_1\mu_2]$$

$$= V(X_1) + V(X_2) + 2\text{Cov}(X_1, X_2)$$

Note: If $X_1 \perp\!\!\!\perp X_2$, then

$$V(X_1 + X_2) = V(X_1) + V(X_2)$$

In general,

$$(a) E\left[\sum_{i=1}^k a_i X_i\right] = \sum_{i=1}^k a_i E(X_i)$$

$$(b) V\left[\sum_{i=1}^k a_i X_i\right] = \sum_{i=1}^k a_i^2 V(X_i) + 2 \sum_{i < j} a_i a_j \text{Cov}(X_i, X_j)$$

(c) If X_1, X_2, \dots, X_k are indep, then

$$V\left[\sum_{i=1}^k a_i X_i\right] = \sum_{i=1}^k a_i^2 V(X_i)$$

EX: Suppose $Y \sim \text{BIN}(n, p)$

Because binomial R.V.'s result from n independent Bernoulli R.V.'s, then

$$Y = \sum_{i=1}^n X_i, \text{ where } X_i \sim \text{BER}(p)$$

$$E[Y] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E(X_i) = \sum_{i=1}^n p = np$$

$$V[Y] = V\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n V(X_i) = \sum_{i=1}^n pq = npq$$

5.3 Correlation

		Y			
		-1	0	1	
X	-1	0	1/4	0	1/4
	0	1/4	0	1/4	1/2
	1	0	1/4	0	1/4
		1/4	1/2	1/4	

Marginal of X & Y:

X	$f_1(x)$	Y	$f_2(y)$
-1	1/4	-1	1/4
0	1/2	0	1/2
1	1/4	1	1/4

$$f(x, Y) = f_x(x) f_y(y)$$

$$f(1, 1) = f_x(1) f_y(1)$$

$$0 \neq \frac{1}{4} \cdot \frac{1}{4}$$

$$\Rightarrow X \not\perp Y$$

$$E(Y) = E(X) = \sum_{\text{all } x} x f_1(x)$$

$$= (-1)\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{4}\right)$$

$$= 0$$

$$E(XY) = \sum_x \sum_y xy f(x, y) = (-1)(-1)(0)$$

$$+ (-1)(0)\left(\frac{1}{4}\right) + (-1)(1)(0) + (0)(-1)\left(\frac{1}{4}\right)$$

$$+ (0)(0)(0) + 0\left(\frac{1}{4}\right)(1) + 1(1)(0)$$

$$+ (1)\left(\frac{1}{4}\right)(0) + 1(1)(0) = 0$$

$$\text{COV}(X, Y) = E(XY) - E(X)E(Y) = 0$$

Remember! If $X \perp\!\!\!\perp Y$, then $\text{Cov}(X, Y) = 0$

But it does not always work the other way around! (The converse is false)

DEF: If X & Y are random variables with variances σ_X^2 & σ_Y^2 and cov σ_{XY} then the correlation coefficient of X & Y is

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

If $\rho = 0$, then X & Y are said to be uncorrelated.

Thm: If ρ is the correlation coeff of X and Y , then $-1 \leq \rho \leq 1$

and $\rho = \pm 1$ iff $Y = aX + b$ with prob 1.

for some $a \neq 0$ and b .