

4.4

Independent R.V.

Table:

$f(x_1, x_2)$		x_2			$f_1(x_1)$
		0	1	2	
x_1	0	.1	.2	.1	.4
	1	.1	.2	.1	.4
	2	.1	.1	0	.2
$f_2(x_2)$.3	.5	.2	

This a pdf since all values

are ≥ 0 and $.1 + .2 + .1 + .1 + .2 + .1 + .1 + .1 = 1$

Note that $P(X_2=1, X_1=0) = P(X_2=1)P(X_1=0)$

$$f(0, 1) = f_2(1) f_1(0)$$

$$.2 = .5 (.4) \checkmark$$

Hence, the events $X_2=1$ & $X_1=0$ are independent events.

However, $X_1=0$ & $X_2=0$ are not independent

Since

$$f(0, 0) \neq f_1(0) \cdot f_2(0)$$

$$.1 \neq .4 (.3)$$

Two variables X_1 & X_2 are independent if

$$f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$$

for all x_1 and x_2

For continuous dist, a similar concept applies

DEF 4.4.1

Independent RV's

Random Var X_1, X_2, \dots, X_K are said to be independent if for every $a_i < b_i$

$$P[a_1 \leq X_1 \leq b_1, a_2 \leq X_2 \leq b_2, \dots, a_K \leq X_K \leq b_K] = \prod_{i=1}^K P[a_i \leq X_i \leq b_i]$$

If the previous holds, the variables are often said to be stochastically independent. If it does not hold, the variables are said to be dependent

Thm 4.4.1 RV X_1, \dots, X_K are independent iff

$$F(x_1, x_2, \dots, x_K) = F_1(x_1) F_2(x_2) \dots F_K(x_K)$$

$$f(x_1, x_2, \dots, x_K) = f_1(x_1) f_2(x_2) \dots f_K(x_K)$$

where $F_i(x_i)$ and $f_i(x_i)$ are the marginal CDFs & pdfs

Thm 4.4.2 Two R.V. X_1 & X_2

with joint pdf $f(x_1, x_2)$ are independent iff

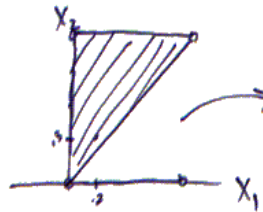
1) The support set

$\{ (x_1, x_2) \mid f(x_1, x_2) > 0 \}$ is a Cartesian product $A \times B$
(The support set is rectangular)

2) The joint pdf can be factored into the product of functions of x_1 & x_2 respectively.

$$f(x_1, x_2) = g(x_1) h(x_2)$$

EX: $f(x_1, x_2) = 8x_1x_2 \quad 0 < x_1 < x_2 < 1$



Support set is not rectangular.

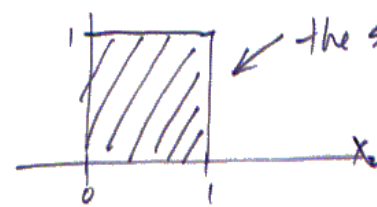
Note: when writing with indicator functions, it is clear that the joint pdf cannot be factored:

$$f(x_1, x_2) = 8x_1x_2 \underbrace{I_{(x_1, 1)}(x_2)}_{\text{function of both } x_1 \text{ \& } x_2 \text{ and cannot be factored}}$$

$$\neq g(x_1) \cdot h(x_2)$$

Hence, x_1 & x_2 are not independent (dependent)

EX: $f(x_1, x_2) = x_1 + x_2 \quad 0 < x_1 < 1, 0 < x_2 < 1$



the support set is rectangular.

With indicator functions:

$$f(x_1, x_2) = (x_1 + x_2) I_{(0,1)}(x_1) I_{(0,1)}(x_2)$$

$$= x_1 I_{(0,1)}(x_1) I_{(0,1)}(x_2) + x_2 I_{(0,1)}(x_1) I_{(0,1)}(x_2)$$

$$\neq g(x_1) \cdot h(x_2)$$

Hence, x_1 & x_2 are not independent.

Note Some consequences of Independence

$$E[XY] = E[X] E[Y]$$

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

Ex: $X \sim \text{GEO}(p)$
 $Y \sim \text{GEO}(p)$ $X \perp\!\!\!\perp Y$

Then the MGF of $T = X + Y$

$$M_T(t) = M_{X+Y}(t) = M_X(t) M_Y(t) \\ = \left(\frac{pe^t}{1 - qe^t} \right) \left(\frac{pe^t}{1 - qe^t} \right) = \left(\frac{pe^t}{1 - qe^t} \right)^2 \\ T \sim \text{NB}(2, p)$$

Note: the expectation of a product is the product of the expectations (if X & Y are independent!)

"Is the integral of a product the product of the integrals?"

$$\int f(x) \cdot g(x) dx = \int f(x) dx \cdot \int g(x) dx? \text{ No!}$$

The rule that applies is the product rule for integrals.

$$\int u dv = uv - \int v du$$

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y) dx dy$$

$$= \left[\int_{-\infty}^{\infty} x f_1(x) dx \right] \left[\int_{-\infty}^{\infty} y f_2(y) dy \right]$$

If X & Y are independent, then $f(x, y) = f_1(x) f_2(y)$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_1(x) f_2(y) dx dy = \int_{-\infty}^{\infty} y f_2(y) \left[\int_{-\infty}^{\infty} x f_1(x) dx \right] dy \\ = \left[\int_{-\infty}^{\infty} x f_1(x) dx \right] \left[\int_{-\infty}^{\infty} y f_2(y) dy \right] = E(X) E(Y)$$

4.5 Conditional Dist.

DEF 4.5.1

Conditional pdf. If X_1 & X_2 are discrete or cont. R.V. with joint pdf $f(x_1, x_2)$, then the cond. pdf of X_2 given $X_1 = x_1$ is

$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

for values such that $f_1(x_1) > 0$, and zero otherwise.

Thm 4.5.1 If X_1 & X_2 are R.V.

with joint pdf $f(x_1, x_2)$ and marginals $f_1(x_1)$ $f_2(x_2)$, then

$$\begin{aligned} f(x_1, x_2) &= f_1(x_1) \cdot f(x_2|x_1) \\ &= f_2(x_2) \cdot f(x_1|x_2) \end{aligned}$$

If X_1 & X_2 are independent, then

$$f(x_2|x_1) = f_2(x_2)$$

$$f(x_1|x_2) = f_1(x_1)$$