

## 4.1 Intro:

In many applications, there are more than one random variable of interest.

$$(X_1, X_2, \dots, X_k)$$

It is convenient to regard these variables as components of a  $k$ -dimensional vector  $X = (X_1, X_2, \dots, X_k)$ , which assumes

values  $x = (x_1, x_2, \dots, x_k)$  in a  $k$ -dimensional

Euclidean space.

The variable may result from repeated measures of  $k$  different characteristics.

## 4.2 JOINT Discrete Dist

DEF: the joint pdf of the  $k$ -dim discrete R.V  $X = (X_1, X_2, \dots, X_k)$  is defined to be

$$f(x_1, x_2, \dots, x_k) = P \left[ \underbrace{X_1 = x_1, X_2 = x_2, \dots, X_k = x_k}_{\text{Note this is an intersection}} \right]$$

for all possible values of  $X$ .

### EX: Extended Hypergeometric Dist

Suppose a collection consists of a finite number of items,  $N$ , and there are  $k+1$  different types of items

$M_1$  are of type 1

$M_2$  are of type 2

$\vdots$

$M_k$  are of type k

$M_{k+1}$  are of type k+1

Then the pdf of  $X$  is

$$f(x_1, x_2, \dots, x_k) = \frac{\binom{M_1}{x_1} \binom{M_2}{x_2} \dots \binom{M_k}{x_k} \binom{M_{k+1}}{x_{k+1}}}{\binom{N}{n}} \quad , 0 \leq x_i \leq M_i$$

$$\text{and } \sum_{i=1}^{k+1} x_i = n \quad \text{and} \quad \sum_{i=1}^{k+1} M_i = N$$

$$\text{Hence } \left. \begin{aligned} x_{k+1} &= n - \sum_{i=1}^k x_i \\ \text{and } M_{k+1} &= N - \sum_{i=1}^k M_i \end{aligned} \right\} X \sim \text{HYP}(n, M_1, \dots, M_k, N)$$

Note: if  $k=1$ , then

$$\frac{\binom{M_1}{x_1} \binom{M_2}{x_2}}{\binom{N}{n}} = \frac{\binom{M_1}{x_1} \binom{N-M_1}{n-x_1}}{\binom{N}{n}} \sim \text{HYP}(N, M_1, n)$$

$$M_2 = N - M_1$$

$$x_2 = n - x_1$$

When selection is with replacement, then we have the multinomial, a generalization of the binomial.

$$X \sim \text{MULT}(n, p_1, p_2, \dots, p_k)$$

numbers distinguishable permutations.

$$f(x_1, x_2, \dots, x_k) = \frac{n!}{x_1! x_2! \dots x_k! x_{k+1}!} p_1^{x_1} p_2^{x_2} \dots p_{k+1}^{x_{k+1}}$$

where  $0 \leq x_i \leq n$

$$x_{k+1} = n - \sum_{i=1}^k x_i$$

$$p_{k+1} = 1 - \sum_{i=1}^k p_i$$

Ex: 4 sided die  
Roll 20 times  
Record the number of occurrences on each side

Prob of obtaining 4 ones, 10 twos, 3 threes, and 3 fours?

$$\begin{aligned} P[X_1=4, X_2=10, X_3=3, X_4=3] &= f(4, 10, 3) \\ &= \frac{20!}{4! 10! 3! 3!} \left(\frac{1}{4}\right)^4 \left(\frac{1}{4}\right)^{10} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^3 \\ &= \frac{775975200}{4^{20}} = \frac{24249225}{34359738368} \\ &= \boxed{0007057} \end{aligned}$$

Thm 4.2.1

$f(x_1, x_2, \dots, x_k)$  is a joint pdf iff

$f(x_1, x_2, \dots, x_k) \geq 0$  and  
 $\sum_{\text{all } x} f(x_1, x_2, \dots, x_k) = 1$

Ex. Two dimensional hypergeometric

Suppose a bin contains 1000 flower seeds

- 400 are red
- 200 are pink
- 400 are white

suppose that 4 seeds are selected at random.

Note: although there are 3 kinds of seeds, only 2 are independent. the last is functionally dependent on the other two.

	White Seeds ( $x_2$ )				
$f(x_1, x_2)$	0	1	2	3	4
0					
1					
2					
3					
4					

$$f(x_1, x_2) = \frac{\binom{400}{x_1} \binom{400}{x_2} \binom{200}{4-x_1-x_2}}{\binom{1000}{4}}$$

What is  $P(1 \text{ white}, 1 \text{ red}, 2 \text{ pink?})$

$$= P(1, 1) = \frac{\binom{400}{1} \binom{400}{1} \binom{200}{2}}{\binom{1000}{4}}$$

$$= .0769$$

$$f(2, 4) = \frac{\binom{400}{2} \binom{400}{4} \binom{200}{-2}}{\binom{1000}{4}} = 0$$

Note: although there are 3 kinds of seeds, only 2 are independent. the last is functionally dependent on the other two.

		White Seeds ( $x_2$ )				
$f(x_1, x_2)$		0	1	2	3	4
Red Seeds ( $x_1$ )	0	.0016	.0127	.0383	.0511	.0254
	1	.0127	.0769	.1541	.1022	0
	2	.0383	.1541	.1539	0	0
	3	.0511	.1022	0	0	0
	4	.0254	0	0	0	0