$\boxed{3.3}$ Special Continuous Dist

We will study several dist.

- Uniform
- Gamma, Beta
- Exponential
- Weibull
- Pareto
- Normal

Uniform

Suppose that a cont. R.V. can assume values only in a bounded interval, say $a < x < b$, and suppose that $f(x)$ is constant over that interval

Then $X \sim UNIF(a,b)$

$$f(x,a,b) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

$$= \frac{1}{b-a} I_{(a,b)}(x)$$

The CDF is

$$F(x;a,b) = \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a<x<b \\ 1, & x \ge b \end{cases}$$

$$= \frac{x-a}{b-a} I_{(a,b)}(x) + I_{[b,\infty)}(x)$$

The expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{2}\frac{b^2-a^2}{b-a}$$

$$= \frac{1}{2}\frac{(a+b)\cancel{(b-a)}}{\cancel{b-a}} = \frac{b-a}{2}$$

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Gamma Distribution

Kappa (Greek letter)

Def 3.3.1 $\boxed{\text{Gamma function}}$

The gamma function, denoted by $\Gamma(\kappa)$ for all $\kappa > 0$, is given by

$$\Gamma(\kappa) = \int_0^\infty t^{\kappa-1} e^{-t}\, dt$$

The gamma function satisfies the following

$$\Gamma(\kappa) = (\kappa-1)\Gamma(\kappa-1), \quad \kappa > 1$$

$$\Gamma(n) = (n-1)!, \quad n = 1, 2, \ldots$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

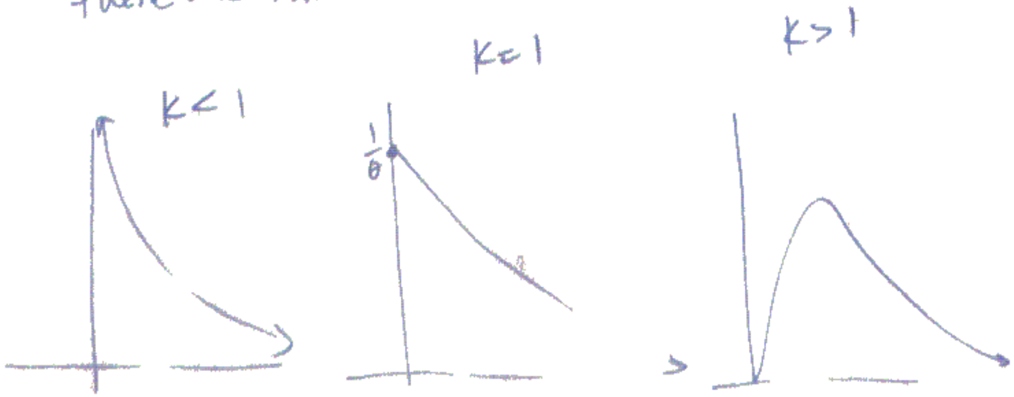
Hint: To prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, write $\left[\Gamma(\frac{1}{2})\right]^2 = \pi$ as a double integral (after subsituting $t = u^2$) and change to polar coordinates.

A contin. R.V. X is said to have the gamma dist with param $\kappa > 0$, and $\theta > 0$ if

$$f(x;\theta,\kappa) = \frac{1}{\theta^\kappa \Gamma(\kappa)} x^{\kappa-1} e^{-\frac{x}{\theta}} I_{(0,\infty)}(x)$$

Note: $X \sim GAM(\theta, \kappa)$

The parameter k is also called a shape parameter because it determines the base shape of the dist. Specifically, there are three different shapes



The CDF of the gamma is

$$F(x,\theta,k) = \left[\int_0^x \frac{1}{\theta^k\Gamma(k)} t^{k-1} e^{t/\theta}\,dt\right] I_{(0,\infty)}(x)$$

The θ parameter is called the Scale parameter. This is important because you don't want your results to depend on scale of measurement used.

A scale parameter satisfies the relation

$$F(x;\theta) = F\left(\frac{x}{\theta}\right)$$

The CDF $F(x;\theta,k)$ cannot be generally solved for. However, when k is an integer, then it can. (This is equiv to doing tab integr

$\boxed{\text{Thm 3.3.2}}$ If $X \sim GAM(\theta, n)$, where n is an integer and if $Y \sim POI\left(\frac{x}{\theta}\right)$, then

$$P[X < x] = P[Y \geq n] = 1 - \sum_{i=0}^{n-1} \frac{(x/\theta)^i}{i!} e^{-x}$$

The mean of the Gamma Dist

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx = \int_0^{\infty} x \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-x/\theta}\,dx$$

$u = \frac{x}{\theta} \Rightarrow x = u\theta$

$du = \frac{1}{\theta} dx \to dx = \theta du$

$$= \int_0^{\infty} u\theta \frac{1}{\cancel{\theta^k}\Gamma(k)} u^{k-1} \cancel{\theta^{k-1}} e^{-u} \cancel{\theta} du$$

$$= \frac{\theta}{\Gamma(k)} \int_0^{\infty} u^k e^{-u} du = \frac{\theta}{\Gamma(k)} \underbrace{\int_0^{\infty} u^{(k+1)-1} e^{-u} du}_{\Gamma(k+1)}$$

$$= \frac{\theta\, \Gamma(k+1)}{\Gamma(k)} = \frac{\theta\, k \cancel{\Gamma(k)}}{\cancel{\Gamma(k)}} = k\theta$$

General Moments:

$$E(X^r) = \int_0^{\infty} x^r \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-x/\theta}\,dx$$

$$= \frac{1}{\theta^k \Gamma(k)} \underbrace{\int_0^{\infty} x^{r+k-1} e^{-x/\theta}\,dx}_{\text{Kernal of Gamma}(\theta, r+k)}$$

$$= \frac{1}{\theta^k \Gamma(k)} \frac{\theta^{k+r}\Gamma(k+r)}{1} \underbrace{\int_0^{\infty} \frac{1}{\theta^{k+r}\Gamma(k+r)} x^{r+k-1} e^{-x/\theta} dx}_{\substack{\text{integrates to } 1 \\ (\text{pdf})}}$$

$$= \frac{\theta^{k+r}\Gamma(k+r)}{\cancel{\theta^k}\Gamma(k)}$$

$$= \frac{\theta^r \Gamma(k+r)}{\Gamma(k)}$$

$\boxed{3.3}$ Cont.

The MGF of Gamma dist is

$$M_X(t) = \left(\frac{1}{1-\theta t}\right)^k$$

Special cases

If $\theta = 2$, $k = \nu/2$, where ν = degrees of freedom,

then $X \sim \chi^2(\nu)$ (chi-squared dist)

Exponential Distribution

If $k=1$, then GAMMA dist is the EXPONENTIAL dist

If $k=1$, then $X \sim EXP(\theta)$

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} I_{(0,\infty)}(x)$$

$$F(x) = \left(1 - e^{-x/\theta}\right) I_{(0,\infty)}(x)$$

Note that θ is a scale parameter.

The exp dist. is useful as a prob. model for lifetimes.

The No-Memory property

Thm 3.3.3. For a cont. R.V. X, $X \sim EXP(\theta)$ iff

$$P[X > a+t \mid X > a] = P[X > t]$$ for all $a > 0$ and $t > 0$.

Think of it as this: An old component that still works is just as reliable as a new compon

<u>Weibull Dist</u>

(A lot like Gamma Dist)

Used for: fatigue & breaking strength of materials
failure times
Engineers love it!
CDF is explicit!

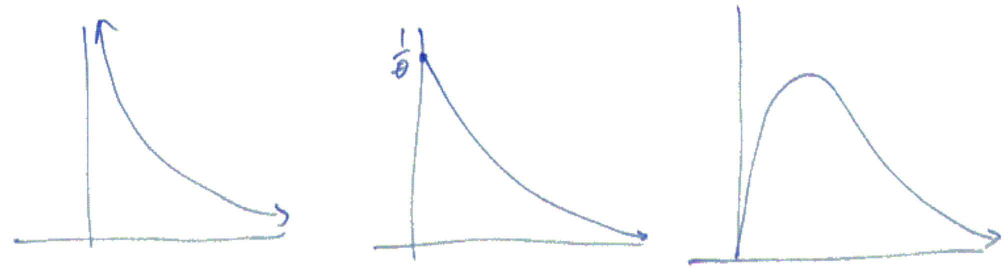
$X \sim WEI(\theta, \beta)$

$$f(x, \beta, \theta) = \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^\beta} I_{(0,\infty)}(x)$$

β = shape parameter
θ = Scale parameter

Shapes:

$\beta < 1$ $\beta = 1$ $\beta > 1$



Note the similarities to the Gamma Dist!

CDF is $F(x, \theta, \beta) = \left(1 - e^{-\left(\frac{x}{\theta}\right)^\beta}\right) I_{(0,\infty)}(x)$

<u>Special cases:</u>

If $\beta = 1$, then $X \sim EXP(\theta)$

If $\beta = 2$, then $X \sim WEI(\theta, 2)$ and we call it the Rayleigh dist

The mean of the Weibull is

$$E(X) = \theta\,\Gamma\left(1+\frac{1}{\beta}\right) \quad \text{valid if } 1+\frac{1}{\beta} > 0$$

$\Rightarrow$ always satisfied since $\beta > 0$

The mgf does not exist in a form that is useful.

The variance is

$$V(X) = \theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right]$$

The $100 \times p^{th}$ percentile is

$$x_p = \theta\left[-\ln(1-p)\right]^{1/\beta}$$

Pareto

$X \sim PAR(\theta, \kappa)$ $\quad$ κ = shape param. $\quad$ θ = scale param.

$$f(x;\theta,\kappa) = \left(\frac{\kappa}{\theta}\right)\left(1+\frac{x}{\theta}\right)^{-(\kappa+1)} I_{(0,\infty)}(x)$$

$\kappa > 0$.

CDF: $F(x;\theta,\kappa) = \left[1-\left(1+\frac{x}{\theta}\right)^{-\kappa}\right] I_{(0,\infty)}(x)$

Useful for modeling length of a wire between flaws: Ex: 2.3.2 is an example.

Note the pdf $f(y) = \left(\frac{\kappa}{a}\right)\left(\frac{y}{a}\right)^{-(\kappa+1)} I_{(a,\infty)}(y)$ is also called a pareto dist.

The mean & variance are

$$E(X) = \frac{\theta}{\kappa-1},\ \kappa>1 \qquad V(X) = \frac{\theta^2\kappa}{(\kappa-2)(\kappa-1)^2},\ \kappa>2$$

p^{th} percentile $\quad X_p = \theta\left[(1-p)^{1/\kappa} - 1\right]$

Normal dist

First published in 1733 as an approx for the sum of Binomial random vars

It is the single most important dist in stats & prob.

$$f(x,\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \sigma > 0$$

Simply it is denoted by $X \sim N(\mu, \sigma^2)$

It is also called the Gaussian dist

Verify it integrates to 1.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2} = \frac{1}{\sqrt{\pi}}\underbrace{\left[2\int_0^{\infty} e^{-u^2}\, du\right]}_{\Gamma(\frac{1}{2})} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$z = \frac{x-\mu}{\sigma}$, $dz = \frac{1}{\sigma}dx$

$u = \frac{t}{\sqrt{2}}$, $du = \frac{1}{\sqrt{2}}dt$

$\phi = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard normal pdf

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt \Big\} \text{Standard normal CDF}$$

Note: $\phi(z) = \phi(-z)$ (Even)

unique max at $z=0$ & inflection pts at ± 1

$\phi'(z) = -z\,\phi(z)$

$\phi''(z) = (z^2-1)\,\phi(z)$

$E(z) = 0$

$E(z^2) = 1 = V(z)$