Normal Dist cont.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\phi'(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{d(z^2/2)}{dz} = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}_{\phi(z)} (-z) = -z\phi(z)$$

$$\phi''(z) = \frac{d(-z\phi(z))}{dz} = -z\phi'(z) + \phi(z)(-1)$$

$$= -z(-z\phi(z)) - \phi(z)$$

$$= (z^2-1)\phi(z)$$

$$z^2\phi(z) = \phi''(z) + \phi(z)$$

$$\phi'''(z) = (z^2-1)\phi'(z) + 2z\,\phi(z)$$

$$= (z^2-1)(-z\phi(z)) + 2z\,\phi(z)$$

$$= (-z^3 + 3z)\phi(z)$$

$$= z(-z^2+3)\phi(z)$$

Max of $\phi(z)$ is $\phi'(z) = 0$

$$-z\phi(z) = 0 \Rightarrow z = 0$$

Inflection pts: $\phi''(z) = 0$

$$(z^2-1)\phi(z) = 0$$

$$z^2 = 1 \Rightarrow z = \pm 1$$

$$E(z) = \int_{-\infty}^{\infty} z\,\phi(z)\,dz = -\int_{-\infty}^{\infty} \phi'(z)\,dz$$

$$= -\lim_{b\to\infty}\Big[\phi(z)\Big]_{-b}^{b} = \lim_{b\to\infty} \phi(-b) - \phi(b)$$

$$= 0 - 0 = 0$$

$$E(z^2) = \int_{-\infty}^{\infty} z^2\phi(z)\,dz = \int_{-\infty}^{\infty} \phi''(z) + \phi(z)\,dz$$

$$= \int_{-\infty}^{\infty} \phi''(z)\,dz + \int_{-\infty}^{\infty} \phi(z)\,dz = \underbrace{\phi'(z)\Big]_{-\infty}^{\infty}} + 1 = 1$$

Thus, $V(Z) = E(Z^2) - [E(Z)]^2$

$= 1 - 0$

$= 1$

Thm 3.3.4

If $X \sim N(\mu, \sigma^2)$, then

1. $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

2. $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

Standard normal cumulative probabilities are in Table 3 in Appendix C in your book

Because of symmetry, only positive z values are given. For negative z-values, $\Phi(-z) = 1 - \Phi(z)$

Ex: $\Phi(z) = .9772$

Let z_γ denote the γ^{th} percentile of the standard normal, which means

$$\Phi(z_\gamma) = \gamma$$

For example, $\Phi(z_{.90}) = .90 \Rightarrow z_{.90} = 1.282$

It is often useful to consider normal prob in terms of standard deviations from the mean.

$$P[\mu - 2\sigma < X < \mu + 2\sigma] = F_X(\mu + 2\sigma) - F_X(\mu - 2\sigma)$$

$$= \Phi\left(\frac{\mu + 2\sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 2\sigma - \mu}{\sigma}\right) = \Phi(2) - \Phi(-2)$$

$= .9772 - (1 - .9772)$

$= .9544$

Thus, $V(Z) = E(Z^2) - [E(Z)]^2$

$= 1 - 0$

$= 1$

$\boxed{\text{Thm } 3.3.4}$

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$= .9772 - (1 - .9772)$

$.9544$

Note that the normal random variable can still be used as a reasonable model for a random variable that takes on only positive values, if very little prob. is associated with the neg. values.
(Another choice is to use a truncated normal model.)

Thm 3.3.5
If $X \sim N(\mu, \sigma^2)$, then

(1) $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$

(2) $E(X-\mu)^{2r} = \dfrac{(2r)!\ \sigma^{2r}}{r!\ 2^r}$, $r = 1, 2, \ldots$

(3) $E(X-\mu)^{2r-1} = 0$, $r = 1, 2, \ldots$

Proof:
(1) Complete the square in the exponent and use the "kernel method".

(2)
(3) Expand the MGF in a Maclaurin series.

$\boxed{3.4}$ Location-Scale Parameters.

$F_0(z)$ represents a completely specified pdf (Does not depend on parameter)

$f_0(z)$ is the pdf corresponding to F_0

$\boxed{\text{DEF}}$

Location Parameters

$\eta \to$ location parameter for a dist X if

$$F(x,\eta) = F_0(x-\eta)$$

$$f(x,\eta) = f_0(x-\eta)$$

Ex $\quad f(x,\eta) = e^{-(x-\eta)} \quad x > \eta$

$$x > \eta \Rightarrow \eta < x < \infty \Rightarrow I_{(\eta,\infty)}(x)$$

$$0 < x-\eta < \infty \Rightarrow I_{(0,\infty)}(x-\eta)$$

Thus, $f(x,\eta) = e^{-(x-\eta)} I_{(0,\infty)}(x-\eta)$

$$f_0(x) = e^{x} I_{(0,\infty)}(x) \Rightarrow f_0(x-\eta) = e^{(x-\eta)} I_{(0,\infty)}(x-\eta)$$

It is common for the location parameter to be a measure of central tendency such as the median or mean (doesn't always happen)

$\boxed{\text{DEF}}$: Scale parameters

$\theta > 0 \Rightarrow$ Scale parameter for a R.V X if

$$F(x;\theta) = F_0\left(\frac{x}{\theta}\right)$$

in other words $\quad f(x,\theta) = f_0\left(\frac{x}{\theta}\right)\left(\frac{1}{\theta}\right) = \frac{1}{\theta} f_0\left(\frac{x}{\theta}\right)$

Ex. $\quad X \sim EXP(\theta) \Rightarrow \theta$ is a scale parameter

$X \sim N(\mu,\sigma^2) \Rightarrow \sigma$ is a scale parameter

Often σ, the std. dev is a scale parameter! not always

$X \sim WEI(\theta,2) \Rightarrow \theta$ is a scale parameter, but it is not the std deviation

DEF 3.4.3

Location Scale Parameters

η and $\theta > 0$ are called location-scale parameters if the CDF has the form

$$F(x, \theta, \eta) = F_0\left(\frac{x \quad \eta}{\theta}\right)$$

or

$$f(x, \theta, \eta) = \frac{1}{\theta} f_0\left(\frac{x-\eta}{\theta}\right)$$

Ex: Cauchy dist.

$$f_0(z) = \frac{1}{\pi} \frac{1}{1+z^2}$$

$$f(x;\theta,\eta) = \frac{1}{\theta} f\left(\frac{x-\eta}{\theta}\right)$$

$$= \frac{1}{\pi\theta} \frac{1}{1+\left(\frac{x-\eta}{\theta}\right)^2}$$

$$X \sim CAU(\theta, \eta)$$

Note: $E(x)$ d.n.e.
$V(x)$ d.n.e

So η & θ do not represent the mean & std dev.