

2.5 Cont

Properties of Moment Generating Functions

Thm 2.5.2

If $Y = aX + b$, then $M_Y(t) = e^{bt} M_X(at)$

Proof.

$$M_Y(t) = E[e^{tY}] = E[e^{t(ax+b)}]$$

$$= E[e^{tb} e^{X(at)}] = e^{tb} \underbrace{E[e^{X(at)}]}_{M_X(at)}$$

$$= e^{tb} M_X(at)$$

It can be shown the MGF's uniquely determine a distribution (if MGF exists)

Application of Thm 2.5.2

$$Y = X - u$$

$$\text{Then } E[(X-u)^r] = \left. \frac{d^r}{dt^r} \left[e^{-ut} M_X(t) \right] \right|_{t=0}$$

Thm 2.5.3 Uniqueness

If X_1 & X_2 have respective CDFs $F_1(x)$ and $F_2(x)$ and MGFs $M_1(t)$ and $M_2(t)$

then $F_1(x) = F_2(x)$ for all real x iff

$M_1(t) = M_2(t)$ for all t in some neighborhood of 0
($-h < t < h$ for some $h > 0$)

Factorial Moments

(DEF 2.5.2)

The r^{th} factorial moment of X is

$E[X(X-1)\cdots(X-r+1)]$ and the factorial moment generating function (FMGF) of X is

$$G_X(t) = E[t^X]$$

If this expectation exists for all t in a neighborhood of 1 ($1-h < t < 1+h$ and $h > 0$)

The factorial moment generating function is also called the prob. generating function because for non-negative integer valued random var. X ,

$$P[X=r] = \frac{G_X^{(r)}(0)}{r!}$$

Note the relationship:

$$G_X(t) = E[t^X] = E[e^{\ln t^X}] = E[e^{(\ln t)X}] = M_X(\ln t)$$

Thm 2.5.4 If X has FMGF, $G_X(t)$, then

$$G'_X(1) = E(X)$$

$$G''_X(1) = E[X(X-1)]$$

⋮

$$G_X^{(n)}(1) = E[X(X-1)\cdots(X-n+1)]$$

Note that regular moments can be derived from the factorial moments. For example:

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$

$$\begin{aligned} \Rightarrow E(X^2) &= E[X(X-1)] + E(X) \\ &= G''_X(1) + G'_X(1) \end{aligned}$$