24 Cont.

Thm 2.4.8 Let $\mu = E(X)$ and $\sigma^2 = Var(X)$

If $\sigma^2 = 0$, then $P[X=\mu] = 1$

Proof. If $x \neq \mu$ for some observed value x, then

$|x-\mu| \geq \frac{1}{i}$ for some integer $i \geq 1$ and

Thus,

$$[X \neq \mu] = \bigcup_{i=1}^{\infty} \left[|X-\mu| \geq \frac{1}{i} \right]$$

Using Boole's Inequality,

$$P[X \neq \mu] \leq \sum_{i=1}^{\infty} \underbrace{P\left[|X-\mu| \geq \frac{1}{i} \right]}_{\text{Chebychev's Inequality.}} \leq \sum_{i=1}^{\infty} i^2 \underbrace{\sigma^2}_{=0} = 0 \Rightarrow P[X \neq \mu] \leq 0$$

$$\Rightarrow P[X \neq \mu] = 0$$

$$P[|X-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

$k\sigma = \frac{1}{i}$

So $P[X=\mu] = 1 - P[X \neq \mu]$

$= 1$

Note that this dist is called a degenerate dist (A dist that concentrates all the prob at one pt

Approximate Mean and Variance

If a function of a Random Var, say $H(X)$ can be expanded in a Taylor series, then an expression for the approx mean and variance can be obtain in terms of the mean & var of X.

DEFine $\mu = E(X)$.

approx equal to

$$(*)\quad H(X) \doteq H(\mu) + H'(\mu)(X-\mu) + \frac{1}{2}H''(\mu)(X-\mu)^2$$

$$E[H(X)] \doteq \underbrace{E[H(\mu)]}_{H(\mu)} + H'(\mu)\underbrace{[E(X) - \underbrace{E(\mu)}_{\mu}]}_{\mu-\mu=0} + \frac{1}{2}H''(\mu)\underbrace{E(X-\mu)^2}_{Var(X)=\sigma^2}$$

$$\boxed{E[H(X)] \doteq H(\mu) + \frac{1}{2}H''(\mu)\sigma^2}$$

$$Var(aX+b) = a^2 Var(X)$$

$$\boxed{Var(H(X)) = Var(\underbrace{H(\mu)}_{b} + \underbrace{H'(\mu)}_{a}(X-\mu)) = [H'(\mu)]^2\sigma^2}$$

The accuracy of these approx depends primarily on the nature of $H(X)$ as well as the amount of variability in X.

[2.5] Moment Generating functions

DEF 2.5.1 If X is a R.V., then the expected value

$$M_X(t) = E[e^{tX}]$$

is called the moment generating function (mgf) of X if the expected value exists for all values of t in some interval of the form $-h<t<h$ for some $h>0$.

Sometimes, the X is dropped and we write $M(t)$ instead of $M_X(t)$

Ex: Assum X is discrete & finite valued.

The mgf is

$$M_X(t) = E[e^{tX}] = \sum_{i=1}^{m} e^{tx_i} f(x_i)$$

The first derivative of $M_X(t)$ wrt t is

$$M_X'(t) = \sum_{i=1}^{m} x_i e^{tx_i} f(x_i)$$

The r^{th} derivative of $M_X(t)$ wrt t is

$$M_X^{(r)}(t) = \sum_{i=1}^{m} (x_i)^r e^{tx_i} f(x_i)$$

Note:

$$M_X^{(r)}(0) = \sum_{i=1}^{m} x_i^r f(x_i) = E[X^r] = \mu_r'$$

Thm 2.5.1 If the mgf exists, then

$$\mu_r' = E(X^r) = M_X^{(r)}(0) \quad \text{for } r = 1, 2, \cdots$$

and

$$M_X(t) = \sum_{r=0}^{\infty} \frac{E(X^r)}{r!} t^r$$

Ex: Consider the cont. R.V. X with pf

$$f(x) = \frac{1}{4} x e^{-x/2}, \quad x > 0$$

The Mgf is

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{1}{4} x e^{-x/2} \, dx$$

$$= \frac{1}{4} \int_0^\infty x e^{(t-\frac{1}{2})x} \, dx = \frac{1}{4} \int_0^{\infty} e^u \frac{du}{t-\frac{1}{2}} \left(\frac{u}{t-1/2}\right)$$

$u = (t-\frac{1}{2})x = tx - \frac{1}{2}x$

$du = (t-\frac{1}{2})dx$

$dx = \frac{du}{\ldots}$

$t < \frac{1}{2}$

$$= \frac{1}{4(t-\frac{1}{2})^2} \int_0^{-\infty} u e^u \, du$$

$u = u \qquad dv = e^u du$

$du = du \qquad v = e^u$

$$= \frac{1}{4(t-\frac{1}{2})^2} \left[u e^u - \int e^u \, du \right]_0^{-\infty}$$

$$= \frac{1}{4(t-\frac{1}{2})^2} \underbrace{\left[u e^u - e^u \right]_0^{-\infty}}_{1}$$

$$= \frac{1}{4\left(\frac{2t-1}{2}\right)^2} = \frac{1}{(2t-1)^2} = \left(\frac{1}{1-2t}\right)^2$$

$, \; t < 1/2$

Ex: A discrete R.V. with pdf

$$f(x) = .7^x, \quad x = 1, 2, \ldots$$

$$M_X(t) = E\left[e^{tx}\right] = \sum_{x=1}^{\infty} e^{tx} (.7)^x$$

$$= \sum_{x=1}^{\infty} (.7e^t)^x = \frac{.7e^t}{1 - .7e^t}$$

$|.7e^t| < 1 \Rightarrow .7e^t < 1$

$e^t < \frac{10}{7}$

$t < \ln\left(\frac{10}{7}\right)$