$\boxed{2.3}$ Cont

Find the median of $F(x) = 1 - e^{-\left(\frac{x}{3}\right)^2}, x > 0.$

Median is where $F(x_{.5}) = .5$

$$1 - e^{-\left(\frac{x_{.5}}{3}\right)^2} = .5$$

$$1 - .5 = e^{-\left(\frac{x_{.5}}{3}\right)^2}$$

$$.5 = e^{-\left(\frac{x_{.5}}{3}\right)^2}$$

$$\ln(.5) = -\left(\frac{x_{.5}}{3}\right)^2$$

$$3\sqrt{-\ln(.5)} = x_{.5} \quad (\text{since } x_{.5} > 0)$$

$$x_{.5} = 3\sqrt{\ln 2} \approx .974 \text{ months}$$

$\boxed{\text{Def:}}$ If the pdf has a unique max at $x = m_0$, (e.g. $\max_{\text{all } x} f(x) = f(m_0)$), then m_0 is called the mode of X

Ex: The pdf of the previous example is

$$f(x) = \left(\frac{2}{9}\right) x e^{-\left(\frac{x}{3}\right)^2}, x > 0$$

$$f'(x) = 0 = \frac{2}{9}\left[x e^{-\left(\frac{x}{3}\right)^2}\left(-\frac{2}{9}x\right) + e^{-\left(\frac{x}{3}\right)^2}\right]$$

$$= \frac{2}{9} e^{-\left(\frac{x}{3}\right)^2}\left[-\frac{2}{9}x^2 + 1\right] = 0$$

$$-\frac{2}{9}x^2 + 1 = 0 \Rightarrow x^2 = \frac{9}{2}$$

$$\boxed{x = \frac{3}{\sqrt{2}} = 2.121 \text{ months}}$$

In general, the mean, median, and mode are all different, but there are cases when they agree.

$\boxed{2.4}$ Some properties of Expected Values

Random Variable X. Consider $Y = u(X)$

(A function of a Random Variable is another random variable.

Thm 2.4.1 If X is a Random variable with pdf $f(x)$ and $u(x)$ is a real valued function whose domain includes all the possible values of X, then

$$E[u(x)] = \begin{cases} \displaystyle\int_{-\infty}^{\infty} u(x) f(x)\, dx, & \text{if } X \text{ is continuous} \\ \displaystyle\sum_{\text{all } x} u(x) f(x), & \text{if } X \text{ is discrete.} \end{cases}$$

$\boxed{\text{Thm } 2.4.2}$ Linearity of the Expectation Operator.

If X is a R.V. with pdf $f(x)$, a, b are constants, and $g(x)$ & $h(x)$ are real-valued functions whose domains include all the possible values of X, then

$$E[a g(x) + b h(x)] = a E g(x) + b E h(x)$$

$$= a E[g(x)] + b E[h(x)]$$

$\boxed{\text{DEF } 2.4.1}$ The variance of a R.V. X is given by

$$\text{var}(X) = E(X - \mu)^2 = E[(X-\mu)^2]$$

Note: σ^2, σ_X^2, and $V(X)$ are all common notations for $\text{var}(X)$. Further $\sigma = \sqrt{\text{var}(X)}$

* σ_X is called the standard deviation.

$\boxed{23}$ cont

DEF A dist with pdf $f(x)$ is said to be symmetric about c if

$$f(c-x) = f(c+x) \quad \text{for all } x.$$

Asymmetric distributions are called skewed dist.

Mixed Distribution

It is possible to have a random variable whose dist. is neither purely discrete nor continuous

A prob dist. for a RV X is of mixed type if the CDF has the form

$$F(x) = aF_d(x) + (1-a)F_c(x)$$

F_d is the CDF of a discrete R.V and F_c is the CDF of a continuous R.V. and $0<a<1$

Corollary

$$Var(X) = E(X^2) - \mu^2, \text{ where } \mu = E(X).$$

Proof: $Var(X) = E(X-\mu)^2$ (by def)

$$= E\left[X^2 - 2X\mu + \mu^2\right]$$

$$= E(X^2) - 2\mu \underbrace{E(X)}_{\mu} + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - \left[E(X)\right]^2$$

Note: The variance (or std. dev) provides a measure of the "spread."

DEF: The k^{th} moment about the orgin of a R.V. X is

$$\mu_k' = E(X^k)$$

and the k^{th} moment about the mean is

$$\mu_k = E\left[(X-\mu)^k\right] = E(X-\mu)^k$$

Note: The mean μ is the first moment about the orgin $\mu = \mu_1'$

The variance σ^2 is the second moment about the mean $\sigma^2 = E(X-\mu)^2 = \mu_2$

$\boxed{\text{Thm 7.4.4}}$ If X is a R.V. and a, b are const. then

$$Var(aX+b) = a^2 Var(X)$$

Note: the mean absolute deviation is defined as

$$E|X-\mu| \qquad (\text{MAD})$$

Thm 2.45 If a dist is symmetric about mean μ, then the third moment about the mean is 0. (e.g. $\mu_3=0$)

$\boxed{\text{Thm. 2.4.6}}$ If X is a R.V. $u(X)$ is a non-neg. real valued function, then for any positive constant $c>0$,

$$P[u(X)\geq c] \leq \frac{E[u(X)]}{c}$$

Markov inequality set $u(x)=|X|^r$, $r>0$

then thm 2.4.6 gives

$$P[|X|\geq c] \leq \frac{E[|X|^r]}{c^r}$$

$\boxed{\text{Thm 2.4.7}}$ Chebychev's Inequality

If X is R.V with mean μ and variance σ^2, then for any $k>0$,

$$P[|X-\mu|\geq k\sigma] \leq \frac{1}{k^2}$$

proof: $u(x)=(X-\mu)^2$

An alternative form is

$$P[|X-\mu| < k\sigma] \geq 1-\frac{1}{k^2}$$