

22 Cont.

DEF. The cumulative dist function (CDF) of a random var X is defined for any real number x by

$$F(x) = P[X \leq x]$$

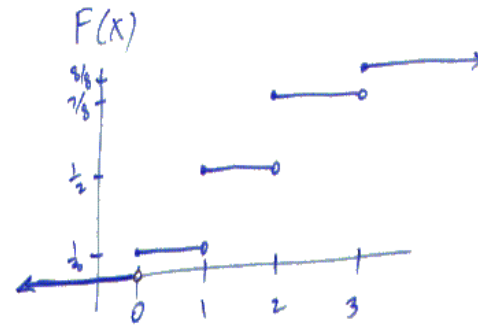
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For example,

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

⇒

x	0	1	2	3
$F(x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8}$



often, we refer to the CDF as the distribution function.

Notation:

$$X \sim f(x) \text{ or } X \sim F(x)$$

means that X has the pdf $f(x)$ and CDF $F(x)$

Note the $F(x)$ is a non decreasing step function

Thm 2.2.2 Let X_i be a discrete R.V. with pdf $f(x)$ and CDF $F(x)$. If the possible values of X are indexed in increasing order

$$x_1 < x_2 < x_3 < \dots < x_n < \dots$$

$$f(x_i) = F(x_i) - F(x_{i-1})$$

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Furthermore, if $x < x_1$, then $F(x) = 0$ and for any other real x ,

$$F(x) = \sum_{x_i \leq x} f(x_i)$$

Thm A function $F(x)$ is a CDF for some R.V. iff it satisfies the following

1. $\lim_{x \rightarrow -\infty} F(x) = 0$
2. $\lim_{x \rightarrow \infty} F(x) = 1$
3. $\lim_{x \rightarrow 0^+} F(x+h) = F(x)$ (Right continuous)
4. If $a < b$, then $F(a) \leq F(b)$

DEF 2.2.3 If X is a discrete R.V. with pdf $f(x)$, then the expected value of X is defined by

$$E(X) = \sum_x X f(x)$$

Expected value is also called the expectation or the mean

Ex: Expected value of
 $X = \#$ of heads in 3 tosses.

X	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\begin{aligned} E(X) &= \sum_X x f(x) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) \\ &= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} \\ &= \frac{12}{8} = 1.5 \end{aligned}$$

Ex: $X = \text{Max}$ of two dice

X	1	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$\begin{aligned} E(X) &= 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right) \\ &= \frac{161}{36} = 4.4722 \end{aligned}$$

2.3 Cont RV

DEF: A random variable X is called a continuous R.V if there is a function $f(x)$, called the pdf of X , such that the CDF can be represented as

$$F(x) = \int_{-\infty}^x f(t) dt$$

Note $f(x) = \frac{d}{dx} F(x) = F'(x)$

Further, Note the def. of $F(x)$

$$F(x) = P[X \leq x]$$

$$\begin{aligned} P[a < X \leq b] &= P[X \leq b] - P[X \leq a] \\ &= F(b) - F(a) \end{aligned}$$

Also,

$$\begin{aligned} P[a \leq X \leq b] &= P(X \leq b) - P(X < a) \\ &= P(X \leq b) - P(X \leq a) + P[X = a] \\ &= F(b) - F(a) + \underbrace{P[X = a]}_{\text{what is this?}} \end{aligned}$$

for a cont. prob dist.,

$$P\{X=c\}=0, \text{ hence,}$$

$$P\{a < X < b\} = P\{a \leq X < b\} = P\{a < X \leq b\} = P\{a \leq X \leq b\}$$

Thm: 2.3.1

A function $f(x)$ is a pdf for some continuous R.V. X iff

1) $f(x) \geq 0$ for all $x \in (-\infty, \infty)$

2) $\int_{-\infty}^{\infty} f(x) dx = 1$

Compare to discrete.

1) $f(x_i) \geq 0$ for all x_i

2) $\sum_{\text{all } i} f(x_i) = 1$

Ex: Suppose $f(x) = \begin{cases} c(1+x)^{-3}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

where c is a constant.

By prop 1, $c > 0$ in order for $f(x) \geq 0$.

By prop 2,

$$1 = \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} c(1+x)^{-3} dx$$

$$= \lim_{b \rightarrow \infty} \left[\frac{c(1+x)^{-2}}{-2} \right]_0^b$$

$$= c \lim_{b \rightarrow \infty} \frac{1}{-2(1+b)^2} + \frac{c}{2}$$

$$= \frac{c}{2} = 1 \Rightarrow \boxed{c=2}$$

Hence

$$f(x) = \begin{cases} 2(1+x)^{-3}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

The cdf for x is

$$F(x) = P\{X \leq x\} = \int_{-\infty}^x f(t) dt$$

Suppose $x \leq 0$, then

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^x 0 dt = 0$$

Suppose $x > 0$, then

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^0 f(t) dt + \int_0^x f(t) dt$$

$$= \int_0^x f(t) dt = \int_0^x 2(1+t)^{-3} dt = \left. \frac{(1+t)^{-2}}{-2} \right|_0^x$$

$$= 1 - \frac{1}{(1+x)^2}$$

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1 - \frac{1}{(1+x)^2}, & x > 0 \end{cases}$$

Note: $f(x)$ is not a probability, although it can be used to assign probability to arbitrarily small intervals.

$$\text{Note: } P(a \leq x \leq b) = \int_a^b f(x) dx = \underset{\substack{\uparrow \\ \text{CDF}}}{F(b) - F(a)}$$

DEF: If X is cont with pdf $f(x)$, the expected value of X is

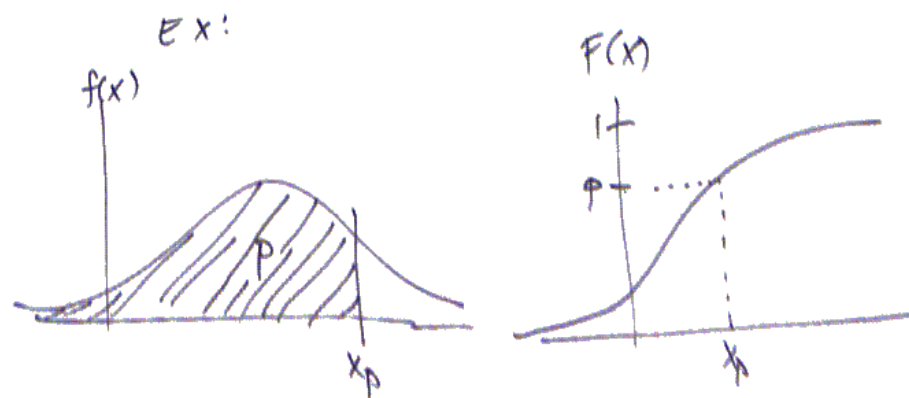
$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

If the integral is absolutely convergent otherwise, we say the $E(X)$ does not exist

DEF 2.3.3

If $0 < p < 1$, then a $100 \times p^{\text{th}}$ percentile of the distribution of a continuous R.V X is a solution x_p to the equation

$$F(x_p) = p$$



We can also think of this in terms of quantiles (eg. the 97th percentile is the 97 quantile)

The median of the dist X is a 50th percentile, denoted by $X_{.5}$ or m .

Ex X is the dist of lifetimes (in months) of a particular component
the CDF is $F(x) = 1 - e^{-(\frac{x}{2})^2}$, $x > 0$

Find the median!