1.6 Counting Techniques

Multiplication Rule

If one operation can be performed n_1 ways
and a second " " " n_2 ",
then there are $n_1 n_2$ ways in which both can be performed together

Ex: How many diff't lunches could a person pick from Bob's Burger joint?

Bob serves 10 diff't burgers
3 sizes of fries
7 flavors of soda

$10 \cdot 3 \cdot 7 = \underline{\underline{210}}$

Thm 1.6.1 If there are N possible outcomes of each of r trials of an experiment, then there are N^r possible outcomes in the sample space.

Proof: (Mult. rule)

$$\underbrace{\underline{N}\ \underline{N}\ \underline{N}\ \underline{N} \cdots \underline{N}\ \underline{N}\ \underline{N}\ \underline{N}\ \underline{N}}_{r \text{ different}} = N^r$$

Ex: How many different ways could a multiple choice test be answered where there is 20 questions & 4 different ~~parts~~ to ~~each~~ question?

$$\underline{4}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{4} \cdots \underline{4}\ \underline{4}\ \underline{4} = 4^{20}$$

$= 1,099,511,627,776$

In counting problems, the order of the selection of objects may or may not be important.

In addition, sometimes sampling is without replacement, or with replacement

Without replacement means an object can be selected at most once. Typically, we are referring to distinct objects (that can be handled)

DEF: An ordered arrangement of n distinguishable objects is known as a permutation

Thm 1.6.2 The number of permutations of n distinguishable objects is $n!$

Note: the term indistinguishable mean that we cannot tell or we don't care about the difference between some objects. Distinguishable means they are all different

$\boxed{\text{Thm 1.6.3}}$ The number of permutations of n objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}$$

Example: □ ○ ◇

Pick 2 objects at a tim

□ ○	○ □	6
□ ◇	◇ □	
○ ◇	◇ ○	

If the order of
then one may be
of combinations
a combination a
of objects. (Y
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En

Ex: □ ○ ◇

□ ○
□ ◇
○ ◇

3 combinations

$$_3C_2 = \frac{3!}{1!\,2!} = \frac{3\cdot\cancel{2\cdot 1}}{\cancel{2\cdot 1}} = 3$$

Thm 1.6.4 The number of combinations of n distinct objects chosen r at a time is

without replacement

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Indistinguishable objects

For example, suppose we want to see how many re-arrangements are possible of the word FOUR. There are 4 different letters, so there is $4! = 24$ different re-arrangements

What about the word SEEN?

It is useful to label the first E as E_1 & second as E_2.

Write out all the possible arrangements using E_1 & E_2 as distinct objects.

$S E_1 E_2 N$	$\cancel{S E_2 E_1 N}$
$S E_1 N E_2$	$\cancel{S E_2 N E_1}$
$S N E_1 E_2$	$\cancel{S N E_2 E_1}$
$E_1 S E_2 N$	$\cancel{E_2 S E_1 N}$
$E_1 S N E_2$	$\cancel{E_2 S N E_1}$
$E_1 N S E_2$	$\cancel{E_2 N S E_1}$
$N S E_1 E_2$	$\cancel{N S E_2 E_1}$
$N E_1 E_2 S$	$\cancel{N E_2 E_1 S}$
$N E_1 S E_2$	$\cancel{N E_2 S E_1}$
$E_1 N E_2 S$	$\cancel{E_2 N E_1 S}$
$E_1 E_2 N S$	$\cancel{E_2 E_1 N S}$
$E_1 E_2 S N$	$\cancel{E_2 E_1 S N}$

Now, drop the E_1 & E_2 to just E.

There are 12 distinct permutations.

Thm 1.6.5 The number of distinguishable permutations of n objects of which r are of one kind and $n-r$ are of the other kind is

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Note: the setting is different here! even though it is the same formula as the combinations rule

Thm: The number of distinguishable permutations of n objects of which r_1 are of one kind, r_2 are of 2^{nd} kind, ⋮ r_k are of k^{th} kind, is

$$\frac{n!}{r_1!\,r_2!\cdots r_k!}$$

Note: $r_1+r_2+\cdots+r_k=n$

Ex: How many different rearrangements of the word MISSISSIPPI are there?

$$\frac{11!}{1!\,4!\,4!\,2!} = 34{,}650$$

Probability

$$P(\text{event } A) = \frac{n(A)}{N}$$

$\leftarrow$ both are computed $\leftarrow$ using counting.

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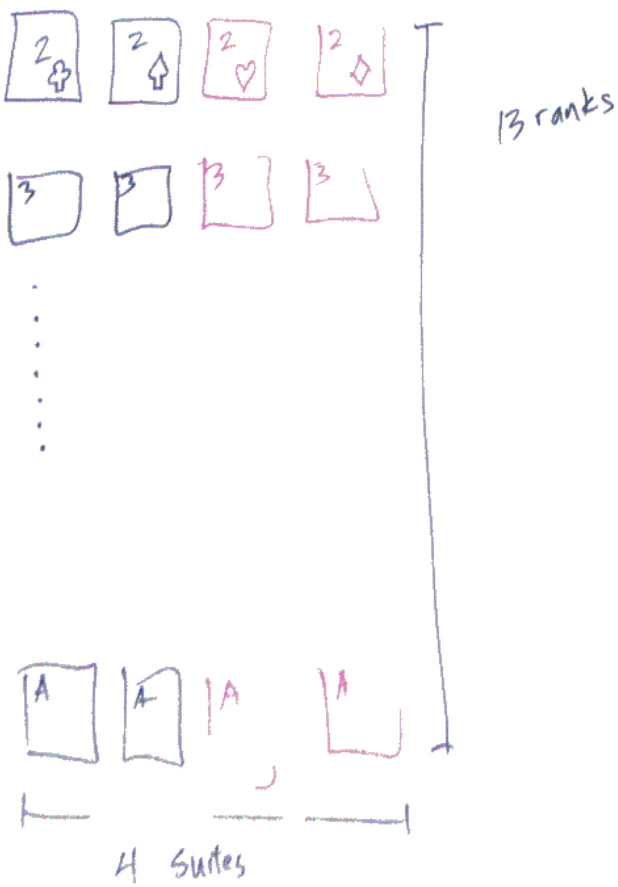

Count the number of Royal Flush → same suite.
└ royalty

A, K, Q, J, 10 — H, D, S, C ⇒ 4 royal flushes.

Straight

2, 3, 4, 5, 6
3, 4, 5, 6, 7
⋮
9, 10, J, Q, K
10, J, Q, K, A

$\underbrace{9 \cdot 4^5}$ this includes straight flushes.

In total, you can pick 52 card 5 at a time:

$$_{52}C_5 = \frac{52!}{5!\,47!}$$

$= 2{,}598{,}960$

⇓

Full-House

3 Kind w/ 2 Kind.

A A A   K K

$13 \cdot {}_4C_3 \cdot 12 \cdot {}_4C_2 = 3{,}744$

↑
choose rank

$$p(\text{full house}) = \frac{3744}{2598960}$$