$\boxed{1.3}$

DEFINITION OF PROB.

$P(A)$ is a set function (That is, it is a function whose domain is a collection of sets, and the range is a subset of the real numbers.

$$P(A) : \underset{\uparrow}{B} \longrightarrow [0, 1]$$

collection of sets.

- Not all set functions are suitable for assigning prob.

<u>DEF</u>: For a given experiment,

S denotes the sample space

$A, A_1, A_2, \ldots$ represent possible events.

A set function $P(A)$ is called the prob of A if the following are satisfied:

$0 \le P(A)$ for every A

$P(S) = 1$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \text{if } A_i \cap A_j = \phi$$

whenever $i \neq j$

First consequence:

$P(\emptyset) = 0$

$S \cup \emptyset = S$

$P(S \cup \emptyset) = P(S)$

$P(S) + P(\emptyset) = P(S)$

$\boxed{P(\emptyset) = 0}$ ✓

Probability in Discrete Spaces

The assignment of prob in the case of discrete spaces can be reduced to assigning probability to elementary events.

Let $\{e_i\}$ be an elementary event.

$$P(\{e_i\}) = p_i$$

So, to satisfy the axioms of prob (def of Prob)

$p_i \geq 0$ for all i.

$$\sum_{\text{all } i} p_i = 1$$

Ex: Coin toss

$S = \{HH, HT, TH, TT\}$

$\Rightarrow$ prob of each event is $1/4$.

$P(HH) = 1/4 = P(HT) =$
$P(TH) = P(TT)$

$X = \#$ of heads in 2 toss

$S = \{0, 1, 2\}$ No!

$\Rightarrow$ $P(\text{each}) = 1/3$??

$P(X=1) = P(HT) + P(TH)$
$= \frac{1}{4} + \frac{1}{4}$
$= 1/2$

$P(0) = P(2) = 1/4.$

Classical Prob

- finite number of outcomes.
- equally likely outcomes

$S = \{e_1, e_2, \cdots, e_N\}$

$P_1 = P_2 = \cdots = P_N = \frac{1}{N}$

any event

$$P(A) = \frac{n(A)}{N} = \frac{\#\text{ of ways } A \text{ can occur}}{\text{total size of sample space.}}$$

Random Selection

If an object is chosen from a finite collection of distinct objects in such a manner that each object has same prob of being chosen, then we say that the object was chosen at random.

1.4 Some properties of Prob.

Thm 1.4.1

If A is an event and A' is its complement, then

$$P(A) = 1 - P(A')$$

Proof: $A \cup A' = S$

$$P(A \cup A') = P(S) = 1$$

$$P(A) + P(A') = 1$$

$$P(A) = 1 - P(A')$$

Ex: $P(\text{At least one girl in 3 births})$

A = At least one girl in 3 births

A' = no girls in 3 births

X = # of girls in 3 births

$$P(A) = P(X=1) + P(X=2) + P(X=3)$$
$$= 1 - P(A')$$
$$= 1 - P(X=0)$$
$$= 1 - \frac{1}{8} = \frac{7}{8}$$

$\boxed{\text{Thm } 1.4.2}$ For any event A, $P(A) \leq 1$

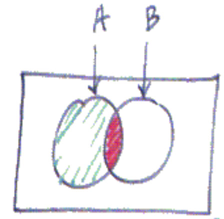
Proof: Not, since $P(A') \geq 0$ (Axiom)

$$1 - P(A) \geq 0 \Rightarrow 1 \geq P(A)$$

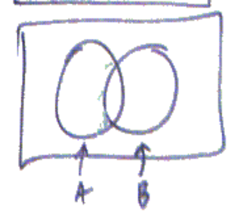
Thm 1.4.3 For any two events A, B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: $A = (A \cap B) \cup (A \cap B')$



$A \cup B = (A \cap B') \cup B$



$P(A) = P(A\cap B) + P(A\cap B')$ (Prob of the union of disjoint events)

$P(A\cap B') \to P(A) - P(A\cap B)$

$P(A\cup B) = P(A\cap B') + P(B)$

$= P(A) - P(A\cap B) + P(B)$

$= P(A) + P(B) - P(A\cap B)$

<u>Thm 1.4.4</u> for any three events

$$P(A\cup \underbrace{B\cup C}_{D}) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

<u>Proof</u>: $P(A\cup D) = P(A) + P(D) - P(A\cap D)$

$= P(A) + P(B\cup C) - P(A\cap (B\cup C))$

$= P(A) + P(B) + P(C) - P(B\cap C) - P((A\cap B)\cup(A\cap C))$

$[P(A\cap B) + P(A\cap C) - P(A\cap B\cap C)]$

<u>Thm 1.4.5</u>

If $A\subset B$, then $P(A)\le P(B)$

<u>Thm 1.4.6</u> Boole's Inequality.

$A_1, A_2, \cdots,$ is a sequence of events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} P(A_i)$$

<u>Thm 1.4.7</u> Bonferroni's Inequality

$$P\left(\bigcap_{i=1}^{k} A_i\right) \ge 1 - \sum_{i=1}^{k} P(A_i')$$

Ex: P(At least one girl in 3 births)

A = At least one girl in 3 births

A' = no girls in 3 births

X = # of girls in 3 births

$P(A) = P(X=1) + P(X=2) + P(X=3)$

$= 1 - P(A')$

$= 1 - P(X=0)$

$= 1 - \frac{1}{8} = \frac{7}{8}$

$\boxed{\text{Thm } 1.4.2}$ For any event A, $P(A) \leq 1$

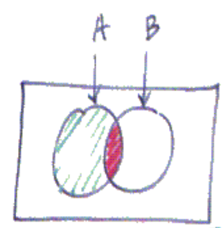
Proof: Not, since $P(A') \geq 0$ (Axiom)

$1 - P(A) \geq 0 \Rightarrow 1 \geq P(A)$

Thm 1.4.3 For any two events A, B

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note: $A = (A \cap B) \cup (A \cap B')$



$A \cup B = (A \cap B') \cup B$