

3.2 Divided Differences

Previous section (Neville's / Lagrange) focused on interpolation at one point.

Divided difference methods generate the polynomials themselves.

Def: n th Lagrange Polynomial in divided diff form.

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

where the constants a_0 are solved for.

We know $P_n(x_k) = f(x_k)$ for $k=0, \dots, n-1$

Let's find a_0 . Plug x_0 into $P_n(x)$ (Note $P_n(x_0) = f(x_0)$ agree at this point)

$$P_n(x_0) = a_0 = f(x_0)$$

Let's find a_1 . Plug x_1 into $P_n(x)$ (Remember $P_n(x_1) = f(x_1)$)

$$P_n(x_1) = a_0 + a_1(x_1 - x_0) = f(x_1)$$

$$f(x_0) + a_1(x_1 - x_0) = f(x_1)$$

$$a_1(x_1 - x_0) = f(x_1) - f(x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let's find a_2 . Plug x_2 into $P_n(x)$ [Remember $P_n(x_2) = f(x_2)$]

$$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$$

At this point, it becomes more difficult (but possible) to do the algebra involved. However, we introduce a new notation:

$$f[x_i] = f(x_i), \quad f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \quad f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

In particular, if we know the divided diff

$f[x_i, x_{i+1}, \dots, x_{i+k-1}]$ and $f[x_{i+1}, x_{i+2}, \dots, x_{i+k}]$, then

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - f[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

With this, we can solve for each of the a_k 's easier.

So, for example, back to the example.

$$P_2(x) = f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

↑ same \Rightarrow see def of a_1 & $f[x_0, x_1]$

$$f(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

↓ notation

$$f[x_2] - f[x_0] - f[x_0, x_1](x_2 - x_0) = a_2(x_2 - x_0)(x_2 - x_1)$$

$$\underbrace{f[x_2] - f[x_1]}_{f[x_1, x_2](x_2 - x_1)} + \underbrace{f[x_1] - f[x_0]}_{f[x_0, x_1](x_1 - x_0)} - f[x_0, x_1](x_2 - x_0) = a_2(x_2 - x_0)(x_2 - x_1)$$

a "trick"

$$f[x_1, x_2](x_2 - x_1) + \underbrace{f[x_0, x_1](x_1 - x_0) - f[x_0, x_1](x_2 - x_0)}_{x_0 \text{'s cancel.}} = a_2(x_2 - x_0)(x_2 - x_1)$$

$$f[x_1, x_2](x_2 - x_1) + f[x_0, x_1](x_1 - x_2) = a_2(x_2 - x_0)(x_2 - x_1)$$

So

$$a_2 = \frac{f[x_1, x_2](x_2 - x_1) + f[x_0, x_1](x_1 - x_2)}{(x_2 - x_0)(x_2 - x_1)} \leftarrow = -(x_2 - x_1)$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]$$

In general,

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$$= f[x_0] + \sum_{k=1}^n f[x_0, x_1, \dots, x_k](x-x_0)\dots(x-x_{k-1})$$

Divided Difference Table

x	$f(x)$	First Divided Differences	Second DD
x_0	$f[x_0]$	$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$	$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$
x_1	$f[x_1]$		
x_2	$f[x_2]$	$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$	$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$
x_3	$f[x_3]$	$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$	$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$
x_4	$f[x_4]$	$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3}$	$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$
x_5	$f[x_5]$	$f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4}$	

You can also define

3 Third DD

2 Fourth DD

1 Fifth DD

Algorithm 3.2 Newton's Interpolatory Divided-Diff Formula

Note that $P(x) = \sum_{i=0}^n F_{i,i} \prod_{j=0}^{i-1} (x-x_j) \Rightarrow$ we're after the numbers on the diagonal. ($F_{i,0} = f(x_i)$)

outputs the polynomial.

Thm 3.6

Suppose $f \in C^n[a,b]$, x_i 's are distinct $\in [a,b]$. $\exists \xi \in [a,b]$

$$f[x_0, x_1, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

Now, reformulate

$$(*) \quad P_n(x) = f[x_0] + \sum_{k=1}^n f[x_0, x_1, \dots, x_k] (x-x_0) \cdots (x-x_{k-1})$$

with equal spacing let $h = x_{i+1} - x_i$ (common for all)

let $x = x_0 + sh$ (point to interpolate)

$$x_i = x_0 + ih \quad \text{and}$$

$$x - x_i = (s-i)h$$

Then (*) becomes

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) \\ + \cdots + f[x_0, x_1, \dots, x_k](x-x_0)(x-x_1) \cdots (x-x_{k-1}) + \cdots + \text{rest one.}$$

$$\text{Note: } (x-x_0)(x-x_1) \cdots (x-x_{k-1}) = (sh)(s-1)h(s-2)h \cdots (s-k+1)h \\ = s(s-1) \cdots (s-k+1)h^k$$

To simplify notation, define an extension of the binomial coefficient

$$\text{as } \binom{s}{k} = \frac{s(s-1) \cdots (s-k+1)}{k!}, \quad \text{where } s \in \mathbb{R} \text{ (any real \#)}$$

Thus,

$$(x-x_0) \cdots (x-x_{k-1}) = \binom{s}{k} k! h^k \quad \text{and}$$

$$P_n(x) = \sum_{k=0}^n f[x_0, \dots, x_k] \binom{s}{k} k! h^k \quad \text{Newton's Forward divided diff formula}$$

If we use Aiken's Δ^2 operator, we can make a simplification to the notation for the formula. First, note

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h} \quad \text{and}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}}{2h} = \frac{\Delta^2 f(x_0)}{2h^2}$$

In general

$$f[x_0, x_1, \dots, x_k] = \frac{1}{k! h^k} \Delta^k f(x_0), \text{ so NFDDM}$$

becomes

$$P_n(x) = \sum_{k=0}^n f[x_0, x_1, \dots, x_k] \binom{s}{k} k! h^k$$

$$P_n(x) = \sum_{k=0}^n \binom{s}{k} \Delta^k f(x_0)$$

Newton
Forward-Difference
formula.

You can also reorder the indices from the back to the front. x_n, x_{n-1}, \dots, x_0 . In this case, we get

Newton backward divided-difference formula

$$P_n(x) = f[x_n] + f[x_{n-1}, x_n](x-x_n) + f[x_{n-2}, x_{n-1}, x_n](x-x_n)(x-x_{n-1}) \\ + \dots + f[x_0, \dots, x_n](x-x_n)(x-x_{n-1}) \dots (x-x_1)$$

Using equal spacing (like before) yields

$$P_n(x) = f[x_n] + s h f[x_{n-1}, x_n] + s(s+1) h^2 f[x_{n-2}, x_{n-1}, x_n] + \dots + s(s+1) \dots (s+n-1) h^n f[x_0, \dots, x_n]$$

DEF: $\nabla P_n \equiv P_n - P_{n-1}$ and

Backwards

Diff operator $\nabla^k P_n = \nabla(\nabla^{k-1} P_n)$

This makes $f[x_{n-1}, x_n] = \frac{1}{h} \nabla f(x_n)$, $f[x_{n-2}, x_{n-1}, x_n] = \frac{1}{2h^2} \nabla^2 f(x_n)$

and, in general, $f[x_{n-k}, \dots, x_n] = \frac{1}{k! h^k} \nabla^k f(x_n)$.

Which means

$$P_n(x) = f[x_n] + s \nabla f(x_n) + \frac{s(s+1)}{2} \nabla^2 f(x_n) + \dots + \frac{s(s+1)\dots(s+n-1)}{n!} \nabla^n f(x_n)$$

Note that the binomial coefficient idea doesn't seem to work because the multiply is going up (not down). But it can still be done using the same extension!

So

$$\binom{-s}{k} = \frac{-s(-s-1)(-s-2)\dots(-s-k+1)}{k!} = \frac{(-1)^k s(s+1)(s+2)\dots(s+k-1)}{k!}$$

Thus, Newton's Backward Difference formula is

$$P_n(x) = \sum_{k=0}^n (-1)^k \binom{-s}{k} \nabla^k f(x_n)$$

Examples

Choose Normal table from 1 to 2 separated by .2