$\boxed{3.1}$ Interpolation and the Lagrange Polynomial

polynomial

<u>EX:</u> suppose we want to find the line that passes through (x_0, y_0) and (x_1, y_1)

then

$$P(x) = ax + b$$

$$y_0 = P(x_0) = a x_0 + b \quad \text{and} \quad y_1 = P(x_1) = a x_1 + b$$

(set b=b)

$$\boxed{\begin{aligned} y_0 &= a x_0 + b \\ y_1 &= a x_1 + b \end{aligned}}$$

system of equations

$$y_0 - a x_0 = y_1 - a x_1$$

$$a x_1 - a x_0 = y_1 - y_0$$

$$a(x_1 - x_0) = y_1 - y_0$$

$$a = \frac{y_1 - y_0}{x_1 - x_0} \implies b = y_1 - a x_1$$

$$= y_1 - \frac{y_1 - y_0}{x_1 - x_0} x_1$$

$$= \frac{y_1(x_1 - x_0) - x_1(y_1 - y_0)}{x_1 - x_0}$$

So $P(x) = ax + b = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)x + b$

This is difficult to extend to more than two points (but it is possible). It requires to solve a $n \times n$ system of equations.

There is an easier way of doing it! Consider the polynomial

$$P(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

Note that this polynomial interpolates the points $(x_0, y_0), (x_1, y_1)$

since $P(x_0) = \left(\frac{x_0 - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x_0 - x_0}{x_1 - x_0}\right) y_1 = y_0$

$P(x_1) = y_1$

The form makes it easy to solve for The polynomial that interpolates.

EX: Find the line that connects $(1,5)$ and $(2,7)$
(x_0, y_0) (x_1, y_1)

$$L_0(x) \cdot y_0 + L_1(x) \cdot y_1$$

So $$P(x) = \frac{(x-2)}{(1-2)}(5) + \frac{(x-1)}{(2-1)}(7)$$

$$= -5(x-2) + 7(x-1)$$

$$= -5x + 10 + 7x - 7$$

$$P(x) = 2x + 3$$

How to extend to higher dimensions?

Let's try 3×3.

We want

$$P(x) = L_{2,0}(x)\, y_0 + L_{2,1}(x)\, y_1 + L_{2,2}(x)\, y_2$$

Goal: If $x = x_0$, then $P(x_0) = y_0 \Rightarrow$ means $L_{2,0}(x_0) = 1$, $L_{2,1}(x_0) = 0$, $L_{2,2}(x_0) = 0$

$\Rightarrow P(x_1) = y_1 \Rightarrow$ means $L_{2,0}(x_1) = 0$, $L_{2,1}(x_1) = 1$, $L_{2,2}(x_1) = 0$

$\Rightarrow P(x_2) = y_2 \Rightarrow$ means $L_{2,0}(x_2) = 0$, $L_{2,1}(x_2) = 0$, $L_{2,2}(x_2) = 1$

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \qquad L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \qquad L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$\boxed{\text{Thm}}$ Lagrange Interpolating Polynomial.

If $x_0, x_1, \ldots, x_n$ are $(n+1)$ distinct numbers

$f(x)$ function whose values are given at each x_i

then there exists a unique polynomial of degree at most n with the property

$$f(x_k) = P(x_k), \text{ where}$$

$$P(x) = f(x_0) L_{n,0}(x) + \cdots + f(x_n) L_{n,n}(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$

where

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

(If it is clear, then the n is dropped from $L_{n,k}(x) = L_k(x)$)

$\boxed{\text{Ex}}$ Find the polynomial that fits $f(x) = 1/x$ between 1 and 4 where we use the points $\rightarrow$ $\begin{cases} f(1) = 1 \\ f(2/3) = 3/2 \\ f(3) = 1/3 \\ f(4) = 1/4 \end{cases}$

So
$$L_0(x) = \frac{(x-2/3)(x-3)(x-4)}{(1-2/3)(1-3)(1-4)} = \frac{(3x-2)(x-4)(x-3)}{6}$$

$$L_1(x) = \frac{(x-1)(x-3)(x-4)}{(\frac{2}{3}-1)(\frac{2}{3}-3)(\frac{2}{3}-4)} = \frac{-27(x-1)(x-3)(x-4)}{70}$$

$$L_2(x) = \frac{(x-1)(x-2/3)(x-4)}{(3-1)(3-2/3)(3-4)} = \frac{-(3x-2)(x-4)(x-1)}{14}$$

$$L_3(x) = \frac{(x-1)(x-2/3)(x-3)}{(4-1)(4-2/3)(4-3)} = \frac{(3x-2)(x-1)(x-3)}{30}$$

So

$$P(x) = 1 \cdot L_0(x) + \frac{3}{2} \cdot L_1(x) + \frac{1}{3} L_2(x) + \frac{1}{4} L_3(x)$$

$$P(x) = -\frac{1}{24}(3x^3 - 26x^2 + 73x - 74)$$

Try $P(2)$ — should be $1/2$ $= -\frac{1}{24}\, 3(2)^3 - 26(2)^2 - 73(2) - 74) = 1/3$

Error bound

If $x_0, x_1, \dots, x_n$ are distinct #'s in $[a,b]$

$f \in C^{n+1}[a,b]$, then

for each $x \in [a,b]$, a number $\xi(x)$ in (a,b) exists

with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Note the similarity to Taylor error term $\dfrac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$

all info concentrated here.

We can use this to approx a complicated function, as long as we know the $(n+1)^{st}$ deriv of $f(x)$

For example, Let use form a partition of the interval from 0 to 1, separated by h. (a constant step size).

Then $x_j = 0 + jh = jh$ and $x_{j+1} = (j+1)h$

What should h be in order to make Linear interpolation be at most off by 10^{-6}? for $f(x) = e^x$ (over $[0,1]$)

So $$|f(x) - P(x)| = \frac{f^{(2)}(\xi)}{2!}|x - x_j||x - x_{j+1}|$$

$$= \frac{f^{(2)}(\xi)}{2}|x - jh||x-(j+1)h| \le \frac{1}{2} e \max_{x_j \le x \le x_{j+1}} |(x-jh)(x-(j+1)h)|$$

Note: If $g(x) = (x-jh)(x-(j+1)h) \Rightarrow$ find $\max|g(x)| \Rightarrow g'(x) \overset{set}{=} 0$

$$g'(x) = (x-jh) + x-(j+1)h = 2x-(2j+1)h = 0 \Rightarrow x = (j+\tfrac{1}{2})h$$

So $$\max_{x_j \le x \le x_{j+1}} |g(x)| = \left|g\left((j+\tfrac{1}{2})h\right)\right| = \left|\left[(j+\tfrac{1}{2})h - jh\right]\left[(j+\tfrac{1}{2})h - (j+1)h\right]\right|$$

$$= \left(\tfrac{1}{2}h\right)\left(-\tfrac{1}{2}h\right) = \frac{h^2}{4}$$

This means that the error is bounded by $|f(x) - P(x)| \le \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}$

Thus, we should choose h such that

$$\frac{eh^2}{8} \le 10^{-6} \Rightarrow h < \sqrt{\frac{8}{e} 10^{-6}} = .0017155$$

One logical choice for h would be $h = .001$
(when $h = .001$, the bound on the error is $.000001$ (10^{-6})

What if you have no idea about the derivs of $f(x)$?
(or you don't know $f(x)$ explicitly (a formula))

n	x	$f(x)$
0	1.0	1
1	1.3	.897470696306
2	1.6	.893515349288
3	1.9	.961765831907
4	2.2	1.101802490888

Suppose we're interested in computing $f(1.45)$ from the table, which would be the best interpreting polynomial to use?

1.0
1.3
1.6
1.9
2.2

P_1 P_2 P_3 P_4

$$P_1(1.45) = \frac{(1.45-1.6)}{(1.3-1.6)}(.8974\ldots) + \frac{(1.45-1.3)}{(1.6-1.3)}(.8935\ldots)$$

$\{2,3\}$

$$= 895493022797$$

$$P_2 = \frac{(1.45-1.6)(1.45-1.9)}{(1.3-1.6)(1.3-1.9)}(.8974\ldots) +$$

use
1.3
1.6
1.9
$\{1,2,3\}$

$$\frac{(1.45-1.3)(1.45-1.9)}{(1.6-1.3)(1.6-1.9)}(.8935\ldots) +$$

$$\frac{(1.45-1.3)(1.45-1.6)}{(1.9-1.3)(1.9-1.6)}(.9617\ldots)$$

$$= .886467294092$$

Others!

Nodes

$\hat{P}_2 = \{0,1,2\} = .883171278213$

$P_3 = \{1,2,3,4\} = .886441065762$

$\hat{P}_3 = \{0,1,2,3\} = .884819286152$

$P_4 = \{0,1,2,3,4\} = .885427453506$

The real answer to $f(1.45) = .68566138027 1$

This gives the following errors for $|P_n(1.45) - f(1.45)|$

Poly	Error
P_1	9.83×10^{-3}
P_2	8.06×10^{-4}
$\hat{P}_2$	2.49×10^{-3}
P_3	7.80×10^{-4}
$\hat{P}_3$	8.42×10^{-4}
P_4	2.34×10^{-4}

Note: The smallest error does not always go with the largest n. Note the difference between P_2 & $\hat{P}_3$

Note:

1) Error term is difficult to apply (you don't always know which is best)
2) If you compute P_2, it doesn't help when computing P_3, P_4, etc

A fix $\Rightarrow$ Neville's Method. We can recursively generate Lagrange polynomial approximations

DEF: Let f be def at $x_0, x_1, \ldots, x_n$. Suppose $m_1, m_2, \ldots, m_k$ are k distinct integers with $0 \le m_i \le k$ $(k \le n)$. The Lagrange poly that agrees with f at those k pts is called $P_{m_1, m_2, \ldots, m_k}$

Ex: Take our example from before

old name	new name		
$P_1 =$	$P_{2,3}$	$P_3 =$	$P_{1,2,3,4}$
$P_2 =$	$P_{1,2,3}$	$\hat{P}_3 =$	$P_{0,1,2,3}$
$\hat{P}_2 =$	$P_{0,1,2}$	$P_4 =$	$P_{0,1,2,3,4}$

$\boxed{\text{Thm 3.5}}$ Let f be def at $x_0, \dots, x_n$

let x_i & x_j be two distinct numb in this set.

Then

$$P(x) = \frac{(x - x_j) P_{0,1,\dots,j-1,j+1,\dots,k}(x) - (x - x_i) P_{0,1,\dots,i-1,i+1,\dots,k}(x)}{(x_i - x_j)}$$

is the k^{th} Lagrange poly that interpolates f at the $k+1$ pts $x_0, x_1, \dots, x_k$. In other words, $P = P_{0,1,2,\dots,k}$

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this implies we can generate the interpolating polyn. recursively

For example, suppose we know $P_{0,1}$ & $P_{1,2}$, then $P_{0,1,2} = \dfrac{(x - x_0) P_{1,2} - (x - x_2) P_{0,1}}{x_0 - x_2}$

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increasing degree of polynomial $\longrightarrow$

Table					
x_0	$P_0 = Q_{0,0}$				
x_1	$P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$			
x_2	$P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2}$		
x_3	$P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$P_{1,2,3} = Q_{3,2}$	$P_{0,1,2,3} = Q_{3,3}$	
x_4	$P_4 = Q_{4,0}$	$\underbrace{P_{3,4}}_{\text{Taken 2 at a time}} = Q_{4,1}$	$P_{2,3,4} = Q_{4,2}$	$P_{1,2,3,4} = Q_{4,3}$	$P_{0,1,2,3,4} = Q_{4,4}$

(Note: $P_i = f(x_i) = y_i$)

note numbers are consecutive

So $Q_{i,j} = P_{i-j,\, i-j+1,\, \dots,\, i}$

Example from our problem.