

2.4 Error Analysis for Iterative Methods

DEF Suppose $\{p_n\}_{n=0}^{\infty}$ is a seq. that converges to p .

If λ & α exist with

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda > 0, \text{ then } \{p_n\} \text{ is}$$

said to converge ^{to p} with order α , with asymptotic error const λ .

In general, a sequence of higher order converges to the solution faster than a seq. of lower order

1. $\alpha = 1 \Rightarrow$ linear convergence
2. $\alpha = 2 \Rightarrow$ quadratic "

EX:

Compare linear to Quadratic

$$\text{Suppose } \lim_{n \rightarrow \infty} \frac{|p_{n+1}|}{|p_n|} = .5 = \lim_{n \rightarrow \infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2}$$

For sufficiently large n ,

$$\frac{|p_{n+1}|}{|p_n|} \approx .5 \approx \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2}$$

For the linear convergent one,

$$|p_n| \approx 0.5|p_{n-1}| \approx 0.5(0.5|p_{n-2}|) \approx 0.5(0.5(0.5|p_{n-3}|)) \approx \dots \approx (0.5)^n |p_0|$$

and for the quadratic conv. one,

$$\begin{aligned} |\tilde{p}_n| &\approx 0.5|\tilde{p}_{n-1}|^2 \approx 0.5[0.5|\tilde{p}_{n-2}|^2]^2 \approx 0.5[0.5[0.5|\tilde{p}_{n-3}|^2]^2]^2 \\ &\approx 0.5^3 |\tilde{p}_{n-2}|^4 \approx 0.5^7 |\tilde{p}_{n-3}|^8 \approx (0.5)^{2^n-1} |\tilde{p}_0|^{2^n} \end{aligned}$$

Suppose $p_n \rightarrow 0$ (linear)
 $\tilde{p}_n \rightarrow 0$ (Quadratic)

Note the table for the first 7 terms

n	P_n (Linear)	\tilde{P}_n (quadratic)
1	5×10^{-1}	5×10^{-1}
2	2.5×10^{-1}	1.25×10^{-1}
3	1.25×10^{-1}	7.8125×10^{-3}
4	6.25×10^{-2}	3.0518×10^{-5}
5	3.125×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

Thm Let $g \in C[a, b]$ and suppose $g(x) \in [a, b] \forall x \in [a, b]$
 Suppose g' is cont on (a, b) and $|g'(x)| \leq k < 1 \forall x \in (a, b)$
 If $g'(p) \neq 0$, then for any P_0 in $[a, b]$, the seq.

$$P_n = g(P_{n-1}), n \geq 1$$

converges only linearly to the unique fixed pt p in $[a, b]$

Proof:

We know from the fixed pt thm, that the seq conv. to p
 since g' exists on $[a, b]$, then, by the MVT

$$P_{n+1} - p = g(P_n) - g(p) = g'(\xi_n)(P_n - p)$$

\uparrow \uparrow \uparrow \uparrow \uparrow
 p is a fixed pt and MVT

where ξ_n is between P_n & p . Since $P_n \rightarrow p$, then $\xi_n \rightarrow p$
 Since g' is cont on $[a, b]$, we have

$$\lim_{n \rightarrow \infty} g'(\xi_n) = g'(p)$$

Thus, $\lim_{n \rightarrow \infty} \frac{P_{n+1} - p}{P_n - p} = \lim_{n \rightarrow \infty} g'(\xi_n) = g'(p)$ and

$$\lim_{n \rightarrow \infty} \frac{|P_{n+1} - p|}{|P_n - p|} = |g'(p)|$$

Hence, if $g'(x) \neq 0$, then P_n converges linearly.

Thm 2.8

Let p be a solution to $x=g(x)$. Suppose $g'(p)=0$ and g' is cont. & strictly bounded by M on an open interval I containing p . Then there exists a $\delta > 0$ such that, for $p_0 \in [p-\delta, p+\delta]$, the seq. defined by $p_n = g(p_{n-1})$ converges at least quadratically to p . Moreover, for sufficiently large values of n ,

$$|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2$$

proof: Choose $\delta > 0$ such that on the int $[p-\delta, p+\delta]$, contained in I , $|g'(x)| \leq K < 1$, and g'' is continuous. Since $|g'(x)| \leq K < 1$, it follows the terms of $\{p_n\}$ are contained in $[p-\delta, p+\delta]$. Expanding $g(x)$ in a Taylor poly about $x=p$ gives

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(\xi)}{2}(x-p)^2, \text{ where } \xi \text{ is between } x \text{ \& } p.$$

Since $g(p) = p$ and $g'(p) = 0$, then

$$g(x) = p + \frac{g''(\xi)}{2}(x-p)^2$$

When $x = p_n$

$$p_{n+1} = g(p_n) = p + \frac{g''(\xi_n)}{2}(p_n - p)^2, \text{ where } \xi_n \text{ is between } p_n \text{ \& } p.$$

Since $|g'(x)| \leq K < 1$ on $[p-\delta, p+\delta]$ and g maps $[p-\delta, p+\delta]$ onto itself, it follows that p_n converges to p (by the fixed pt thm)

Since ξ_n is between p_n & p , and $p_n \rightarrow p$, then $\xi_n \rightarrow p$ also

and

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(p)|}{2} \Rightarrow \text{this implies quad convergence.}$$

Since g'' is cont & bounded by M , then $|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2$

Thm 2.7 & 2.8 say to choose a $g(x)$ so that $g'(p) = 0$

Suppose $g(x) = x - \phi(x)f(x)$. Let's find what $\phi(x)$ needs to be to make $g'(p) = 0$ deriv is zero at the fixed pt.

$$g'(x) = 1 - \phi'(x)f(x) - \phi(x)f'(x)$$

$$g'(p) = 1 - \phi'(p)\underbrace{f(p)}_{=0} - \phi(p)f'(p)$$

↓ force

$$0 = 1 - \phi(p)f'(p) \Rightarrow \phi(p) = \frac{1}{f'(p)}$$

So

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p)}$$

Since p is usually unknown, then let $p = p_{n-1}$ and

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad (\text{Newton's method!})$$

Start with converge to zero of $f(x) = x^2 - 2x + 1$ example.

DEF: A solution p of $f(x) = 0$ is said to be a zero of multiplicity m if f can be written as $f(x) = (x-p)^m g(x)$, for $x \neq p$, where $\lim_{x \rightarrow p} g(x) \neq 0$.

Thm 2.10 $f \in C^1[a, b]$ has a simple zero at p in (a, b) iff $f(p) = 0$, but $f'(p) \neq 0$.

Thm 2.11 The function $f \in C^m[a, b]$ has a zero of mult m at p iff $0 = f(p) = f'(p) = \dots = f^{(m-1)}(p)$, but $f^{(m)}(p) \neq 0$.

EX: $f(x) = 2\cos x - 2 - x^2$ has a zero of mult 2 at 0

Note: $f'(x) = -2\sin x - 2x$

$$f''(x) = -2\cos x - 2$$

$$f(0) = 2\cos 0 - 2 - 0 = 2 - 2 = 0$$

$$f'(0) = -2\sin 0 - 2(0) = 0 - 0 = 0$$

$$f''(0) = -2(1) - 2 = -4 \neq 0$$

Modified Newton's Method

Newton's method does not converge quadratically when p is a zero of mult 2 or larger. In this case, we develop a modified function. Suppose f has a zero of mult. m at p . So $f(x) = (x-p)^m g(x)$

$$u(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m g(x)}{m(x-p)^{m-1} g(x) + (x-p)^m g'(x)}$$

$$= \frac{(x-p) g(x)}{m g(x) + (x-p) g'(x)} \Rightarrow \text{this has a root at } p, \text{ but it is a simple root.}$$

So apply $f(x) = 0$ to $u(x) = 0$

$$\text{Let } g(x) = x - \frac{u(x)}{u'(x)} = x - \frac{f(x)/f'(x)}{\frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}}$$

$$= x - \frac{f(x)}{f'(x)} \cdot \frac{[f'(x)]^2}{[f'(x)]^2 - f(x)f''(x)}$$

$$= x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

So

$$x_{n+1} = x_n - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

Modified Newton's method.