2.3

Newton's Method

Related to fixed pt iteration (a faster one) will cover next section

Derivation using Taylor's Thm

Consider the first Taylor Polynomial. Let $\hat{x} \in [a,b]$ be an approx to the zero, p, such that $f'(\hat{x}) \neq 0$ and $|\hat{x} - p|$ is small

Then

$$f(x) = f(\hat{x}) + f'(\hat{x})(x-\hat{x}) + \frac{f''(\xi(x))(x-\hat{x})^2}{2}, \quad \xi \text{ is between } x \text{ and } \hat{x}$$

Since $f(p)=0$, then

$$f(p) = 0 = f(\hat{x}) + f'(\hat{x})(p-\hat{x}) + \frac{f''(\xi(p))(p-\hat{x})^2}{2}$$

If we assume that $|p-\hat{x}|$ is small, then $(p-\hat{x})^2$ is negligible and

$$0 \approx f(\hat{x}) + f'(\hat{x})(p-\hat{x})$$

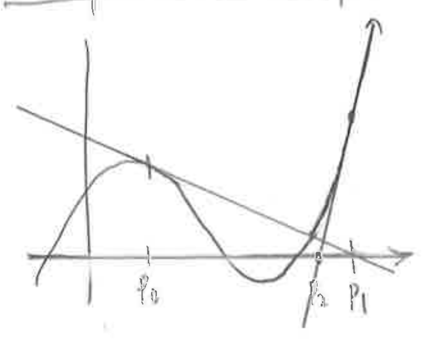
Solving for p yields:
$$p \approx \tilde{x} - \frac{f(\tilde{x})}{f'(\tilde{x})}$$

If we use the above as an iteration that generates a seq. $\{p_n\}$ then $p = p_n$, $\hat{x} = p_{n-1}$ and

$$p_n = p_{n-1} + \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Graphical Explanation

(use R for a good pic)



What stopping technique should we use?

We could use

$|p_N - p_{N-1}| < \varepsilon$ (abs error)

or $\dfrac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon$ (rel error)

or $|f(p_N)| < \varepsilon$ (found a zero)

which is best? (see bisection for a discussion they all have issues!)

We should also have a limit to the number of iterations to prevent unterminating loop.

Examples

EX: $f(x) = \cos x - x = 0 \Rightarrow p_n = p_{n-1} - \dfrac{\cos p_n - p_n}{-\sin p_n - 1}, \; n \geq 1$

$f'(x) = -\sin x - 1$

This only requires 4 iterations if we start with $p_0 = \pi/4$

EX: from last section:

$f(x) = x^3 + 4x^2 - 10 = 0 \qquad p_n = p_{n-1} - \dfrac{p_{n-1}^3 + 4p_n^2 - 10}{3p_n^2 + 8p_n}, \; n \geq 1$

$f'(x) = 3x^2 + 8x$

This only requires 3 iterations from $p_0 = 1.5$ to get 8 decimals correct

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EX: Square Root Alg.

$f(x) = x^2 - m$

$f'(x) = 2x$

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - m}{2x_n} = \frac{2x_n^2 - x_n^2 + m}{2x_n}$$

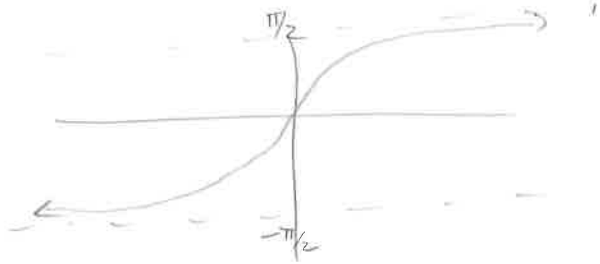
$$= \frac{x_n^2 + m}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{m}{x_n}\right)$$

To find a square root, all you need is multiplication, division, and summation!!

Ex: $f(x) = \tan^{-1} x$ $\Longrightarrow$

$f'(x) = \dfrac{1}{1+x^2}$



$$x_{n+1} = x_n - \frac{\tan^{-1} x_n}{1+x_n^2}$$

Try it with $x_0 = 1.5$

$x_1 = -1.69408$
$x_2 = 2.321127$
$x_3 = -5.11409$
$x_4 = 32.2956839$
$x_5 = -1575.31$
$\vdots$
$x_{11} = 2.45399 \times 10^{108}$

R code

```
f = function(x){ atan(x)}
fp= funtion(x){ 1/(1+x^2)}
x=x-f(x)/fp(x); x
```

Try with $x_0 = 1.3$

$x_1 = -1.1616$
$x_2 = 0.858896$
$x_3 = -0.3742407$
$x_4 = 0.03401887$
$x_5 = -.0000262$
$x_6 = .000000000000012045$

Why the difference? Newton's Method is not good if you don't have a good initial guess!

Another case $f(x) = x^2 - 4x + 4$ takes 25 iterations to converge 7 dec digits

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Bisection is better! (Robust, but slow!)
Newton's Method is better! (fast! but fails too!)

$\boxed{\text{Thm } 2.6}$

Let $f \in C^2[a,b]$. If $p \in [a,b] \ni f(p) = 0$ and $f'(p) \neq 0$
Then there exists a $\delta > 0$ such that Newton's Method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approx $p_0 \in [p-\delta, p+\delta]$

proof: show that $g(x)$ satisfies the fixed pt Thm where $g(x) = x - \dfrac{f(x)}{f'(x)}$

$$\left(|g'(x)| \le k < 1 \quad \text{and} \quad g: [p-\delta, p+\delta] \to [p-\delta, p+\delta] \right)$$

As long as the initial guess is close enough, it will converge (and that $f'(p) \neq 0$)

Newton's Method major difficulty = you have to know $f'(x)$.

For example, suppose $f(x) = x^2 3^x \cos 2x$ $f'(x)$ is difficult.
We can circumvent this problem by approx the derivative.

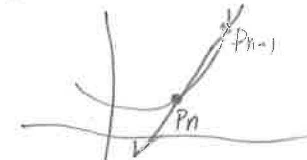
SECANT METHOD

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Newton's Method

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

We plan to use the last two iterations to figure the estimate for the slope.



So let $x_0 = p_{n-1}$ and $x = p_{n-2}$. Thus,

$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}} \quad \text{(either one)}$$

Thus,

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{f(p_{n-1})}{\frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}}$$

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}}$$

Ex: $f(x) = x^2 - 3$

$$\left.\begin{aligned} p_0 &= 1 \\ p_1 &= 2 \end{aligned}\right\} \text{initial guesses}$$

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{f(2)(2-1)}{f(2) - f(1)} = 2 - \frac{(1)(1)}{1 - (-2)} = 2 - \frac{1}{3} = 5/3$$

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = \frac{5}{3} - \frac{f(5/3)(\frac{5}{3} - 2)}{f(5/3) - f(2)} = \frac{5}{3} - \frac{(-2/9)(-1/3)}{(-2/9) - 1} = \frac{5}{3} + \frac{2}{33} = \frac{19}{11}$$

Note: on each step two iterations should be kept.

Alg Input guesses p_0 & p_1

Step1 Let $i=2$

$fp0 = f(p_0)$

$fp1 = f(p_1)$

Step2 while $i \le N_0$ do Steps 3-6

Step3 $p = p_1 - \dfrac{fp1(p_1 - p_0)}{fp1 - fp0}$

Step4 If $|p-p_1| < TOL$, then output (p); Stop!

Step5 Let $i = i+1$

Step6 (update p_0, p_1)

$p_0 = p_1$; $fp0 = f(p_0)$

$p_1 = p$; $fp1 = f(p_1)$

Step7 Output "Method failed"

Compared to Newton's Method:

1) Not as fast (Not so good!)
2) does not require $f'(x)$ evaluation (Good!)
3) Each step only requires one f evaluation. (Good!)

Secant and Newton's are used to refine the answer obtained from another method, like Bisection)

Bisection is great in that the root is always between successive iterations! This isn't guaranteed with Newton's Method & Secant Method.

We can modify Secant Method so the root is always bracketed like Bisection. $[p_0, p_1]$ $(p_0 < p_1)$

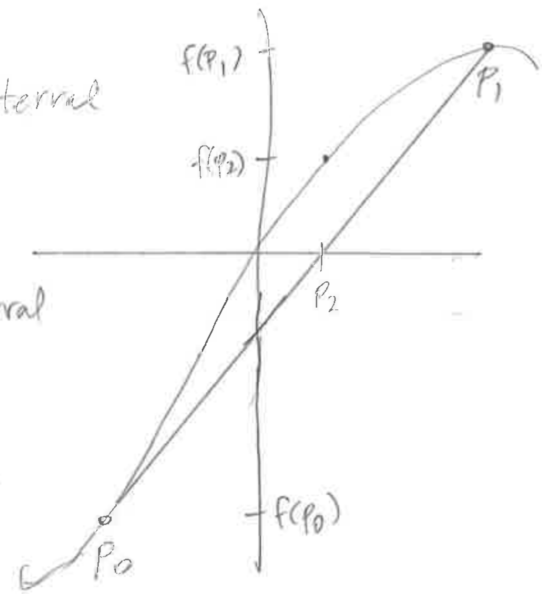
So, given two points, p_0, p_1, we find p_2 using secant, but then check if $f(p_2)$ has the same sign as $f(p_0)$ or $f(p_1)$

If it is different as $f(p_0)$, then

use $[p_0, p_2]$ as the next interval

If the sign is the same than $f(p_0)$, then

Use $[p_2, p_1]$ as the next interval



On each step, the root is bracketed.
This method is called False Position.
Note on the Alg. that only Step 6 & 7 are different.

guarantee convergence! Worst case is <u>linear</u>. can be fixed. (Illinois Alg.)