

2.2

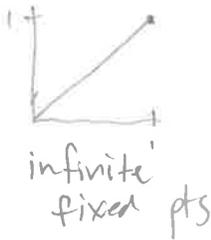
Fixed-Point Iteration

A fixed point for a function g is a number p for which $g(p) = p$

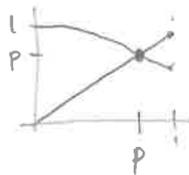
Root finding is equivalent: $\underbrace{g(p) - p}_{f(p)} = 0$

Picking the right $g(p)$ function for a given $f(x)$ can sometimes lead to powerful root finding techniques.

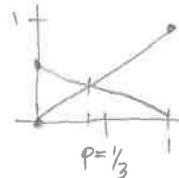
EX: $g(x) = x$



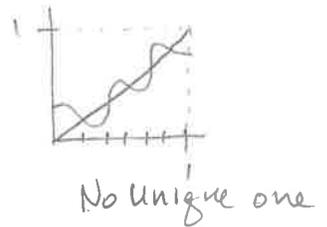
$g(x) = \cos x - x$



$g(x) = \frac{1}{2}x + \frac{1}{2}$



$g(x) =$



How do we get a unique one over any particular $[a, b]$?

Sufficient conditions for uniqueness & existence of a fixed pt:

Thm 2.1 If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$ (both x & y are in the box)

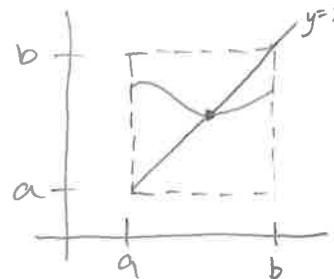
then g has a fixed point in $[a, b]$

Suppose that $g \in C'(a, b)$ and

$\exists 0 < k < 1$ with

$|g'(x)| \leq k < 1 \quad \forall x \in (a, b)$

Then the fixed pt is unique in $[a, b]$



starts at left "wall" and ends at right "wall" and slope isn't too big!

proof: Next

Proof: If $g(a)=a$ or $g(b)=b$, then the fixed pt exists.

Suppose $g(a) \neq a$ and $g(b) \neq b$, then
 $g(a) > a$ and $g(b) < b$.

Define $h(x) = g(x) - x$. It follows that

h is cont on $[a, b]$ and

$$h(a) = g(a) - a > 0$$

$$h(b) = g(b) - b < 0$$

By the Intermediate Value Thm, $\exists p \in (a, b) \ni h(p) = 0$

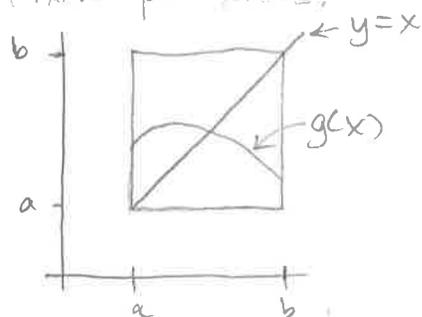
Thus, $h(p) = 0 = g(p) - p \Rightarrow g(p) = p$. (Fixed point exists!)

Next, suppose $|g'(x)| \leq k < 1$ holds and $p \neq q$ are both fixed pts in $[a, b]$ where $p \neq q$.

By MVT, $\exists \xi$ between $p \neq q$ (and thus in $[a, b]$) with

$$\frac{g(p) - g(q)}{p - q} = g'(\xi) \Rightarrow |p - q| = |g(p) - g(q)| = |g'(\xi)| |p - q| \leq k |p - q| < |p - q|$$

so $|p - q| < |p - q| \Rightarrow$ impossible! It must be $p = q$.



Examples

$g(x) = \ln\left(\frac{7}{x}\right)$ on $[1, 2]$. Using EVT, $g'(x) = \frac{-1}{x} \neq 0$ on $[1, 2]$

It follows that g is 1-1 on $[1, 2]$ and will have max values at the endpts:

$$\begin{aligned} g(1) &= \ln 7 \approx 1.946 \\ g(2) &= \ln \frac{7}{2} \approx 1.253 \end{aligned} \Rightarrow g(x) \in [1.253, 1.946] \subset [1, 2]$$

Thus, a fixed pt exists in $[1, 2]$

Further, $g'(x)$ is 1-1 function on $[1, 2]$ and obtains max on endpts. Thus, $g'(1) = -1$, $g'(2) = -\frac{1}{2} \Rightarrow |g'(x)| \leq 1$

$k=1 \Rightarrow$ This seems to fail. But we can move the left pt of the interval. So choose $[1.5, 2]$ instead. \Rightarrow It still satisfies the first part, and now $|g'(1.5)| \leq \frac{2}{3} = k < 1$

It is also possible to have a fixed pt, but not have the conditions of thm 2.1 satisfied. See book.

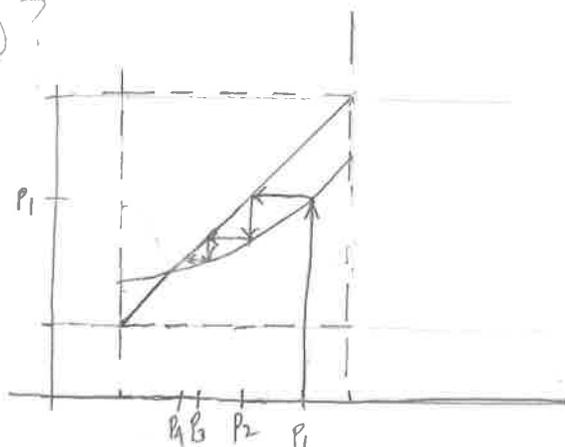
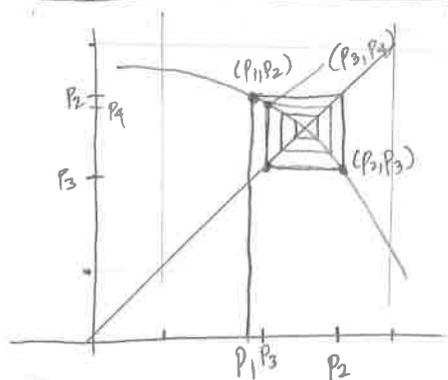
To approx a fixed pt, we choose an initial approx p_0 and then generate a sequence $\{p_n\}_{n=0}^{\infty}$ by letting

$$p_n = g(p_{n-1}) \quad \text{for each } n \geq 1$$

If the sequence converges and g is cont, then

$$p = \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} g(p_{n-1}) = g(\lim_{n \rightarrow \infty} p_{n-1}) = g(p)$$

How does it work visually?



How do you choose $g(p)$ for $f(x)=0$?

Suppose we want to solve $x^3 + 4x^2 - 10 = 0$

It has a root in $[1, 2]$ (by IVT)

We could do:

$$x^3 + 4x^2 - 10 = 0 \Rightarrow 4x^2 = 10 - x^3$$

$$x = \frac{\sqrt{10 - x^3}}{2} \quad (\text{choose positive})$$

$$\text{Let's call this one } g_3(x) = \frac{\sqrt{10 - x^3}}{2}$$

Here's some others!

$$\begin{aligned} x^3 + 4x^2 - 10 = 0 &\Rightarrow g(x) = x - f(x) \\ g_1(x) &= x - x^3 - 4x^2 + 10 \end{aligned}$$

$$x^3 + 4x^2 - 10 = 0$$

$$x^3 = 10 - 4x^2$$

$$x^2 = \frac{10 - 4x^2}{x} \Rightarrow x = \sqrt{\frac{10 - 4x^2}{x}} = g_2(x) \quad (\text{choose pos})$$

$$x^3 + 4x^2 - 10 = 0$$

$$x^2(x+4) = 10$$

$$x^2 = \frac{10}{x+4}$$

$$x = \sqrt{\frac{10}{x+4}} = g_4(x)$$

$$g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

$$= \frac{2x^3 + 4x^2 + 10}{3x^2 + 8x}$$

Results:

$$g_1(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \Rightarrow \text{fails to converge}$$

$$g_2(x) = \sqrt{\frac{10 - 4x^2}{x}} \Rightarrow \text{fails to converge}$$

$$g_3(x) = \frac{\sqrt{10 - x^3}}{2} \Rightarrow \text{converges in 30 iterations}$$

$$g_4(x) = \sqrt{\frac{10}{x+4}} \Rightarrow \text{converges in 15 iterations}$$

$$g_5(x) = \frac{2x^3 + 4x^2 - 10}{3x^2 + 8x} \Rightarrow \text{converges in 4 iterations!}$$

Note that it is difficult to tell which converge by sight. How can we determine which will converge and how rapidly?

Thm 2.3 (Fixed pt thm)

Let $g \in C[a, b]$

$$g(x) \in [a, b] \quad \forall x \in [a, b]$$

Suppose $g'(x)$ exists on (a, b) and

$$|g'(x)| \leq K < 1 \quad \text{for all } x \in (a, b)$$

If p_0 is any number in $[a, b]$, then $\{p_n\}_{n=0}^{\infty}$ converges to the unique fixed pt in $[a, b]$ $p_n = g(p_{n-1})$

proof: By Thm 2.2, a unique fixed pt exists. $p_n \in [a, b]$

By the MVT, we have

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi)| |p_{n-1} - p| < K |p_{n-1} - p|, \quad \text{where } \xi \in (a, b)$$

$$\text{So } |p_n - p| \leq K |p_{n-1} - p| \leq K^2 |p_{n-2} - p| \leq \dots \leq K^n |p_0 - p|$$

$$\text{Since } K < 1, \text{ then } \lim_{n \rightarrow \infty} |p_n - p| \leq \lim_{n \rightarrow \infty} K^n |p_0 - p| = 0 \Rightarrow p_n \rightarrow p \text{ as } n \rightarrow \infty$$

Corollary Bound for the error

If g satisfies the hypothesis for Thm 2.3, then the bound for the error involve using p_n to approx p are given by:

$$|p_n - p| \leq K^n \max\{p_0 - a, b - p_0\}$$

$$\text{and } |p_n - p| \leq \frac{K^n}{1-K} |p_0 - p|$$

proof: last inequality from previous proof:

$$\begin{aligned} |p_n - p| &\leq K^n |p_0 - p| \rightarrow \begin{array}{c} | \\ \hline a \quad p_0 \quad p \quad b \end{array} \\ &\leq K^n \max\{p_0 - a, b - p_0\} \end{aligned}$$

Using the MVT, it follows that

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq K |p_n - p_{n-1}| < \dots < K^n |p_1 - p_0|$$

Thus, for $1 \leq n < m$

$$\begin{aligned} |P_m - P_n| &= |P_m - P_{m-1} + P_{m-1} - P_{m-2} + \dots - P_{n+1} + P_{n+1} - P_n| \\ &\leq |P_m - P_{m-1}| + |P_{m-1} - P_{m-2}| + \dots + |P_{n+2} - P_{n+1}| + |P_{n+1} - P_n| \\ &\leq K^{m-1} |P_1 - P_0| + K^{m-2} |P_1 - P_0| + \dots + K^{n+1} |P_1 - P_0| + K^n |P_1 - P_0| \\ &= K^n |P_1 - P_0| \underbrace{(1 + K + K^2 + \dots + K^{m-n-1})}_{\text{geometric sum}} \end{aligned}$$

So, let $m \rightarrow \infty$

$$\lim_{n \rightarrow \infty} |P_m - P_n| = |P - P_n| \leq \lim_{m \rightarrow \infty} K^n |P_1 - P_0| \underbrace{\sum_{i=0}^{\infty} K^i}_{\frac{1}{1-K}} = \frac{K^n}{1-K} |P_1 - P_0|$$

Thus, the smaller the K , the faster the convergence!

What about the choices for g_1, g_2, g_3, g_4, g_5 ?

$$g_1(x) = x - x^3 - 4x^2 + 10$$

$$g_1'(x) = 1 - 3x^2 - 8x$$

No interval containing P for which $g_1'(x) < 1$

move on

$$g_2(x) = \left(\frac{10 - 4x^2}{x} \right)^{1/2}$$

Bad why?

doesn't map $[1, 2]$ onto $[1, 2]$

also $g_2'(P) = 3.4 > 1$ (K too big)

$g_3(x) \Rightarrow$ Note $[1, 2]$ fails
but $[1, 1.5]$ works

$$g_3([1, 1.5]) \subset [1, 1.5]$$

$$g_3'(x) \leq .666$$

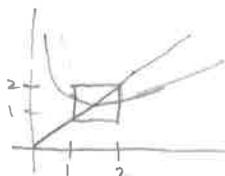
$$g_4(x) = \sqrt{\frac{10}{4+x}}$$

$$g_4'(x) = \left| \frac{-5}{\sqrt{10}(4+x)^{3/2}} \right| \leq \frac{5}{\sqrt{10} \cdot 5^{3/2}} = \frac{\sqrt{2}}{10} \approx .14$$

should converge fast!

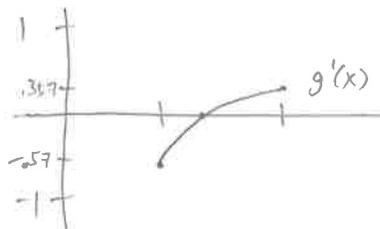
What about g_5 ?

$$g(1) = 1.45, g(2) = 1.5$$



$$g'(x) = \frac{2(3x-4)(x^3+4x^2-10)}{x^2(3x+8)^2} \Rightarrow \text{roots are } 1.365, 4/3$$

$$\left. \begin{aligned} g'(1) &= \frac{-70}{121} \\ g'(4/3) &= \frac{-7}{216} \\ g'(1.365) &= 0 \\ g'(2) &= \frac{5}{14} \end{aligned} \right\} |\max| = .5785 = K$$



Use $p_0 = 1$, and $p_1 = \frac{16}{11} = 1.4545$

By thm 2.3 corollary,

$$|p_n - p| \leq \frac{K^n}{1-K} |p_1 - p_0| = \frac{(.5785)^n}{1-.5785} \underbrace{||1.4545 - 1|}_{\frac{5}{11}} = \frac{55}{51} \left(\frac{70}{121}\right)^n$$

To be guaranteed 10 digits, $\frac{55}{51} \left(\frac{70}{121}\right)^n = 10^{-8}$ $n = 33.7$
 $= 34.$

This still converges faster! Only 4 are needed.

Note that it is not really that close to .57 very long. Close to

$1.33 = \frac{4}{3}$, the value is $\frac{7}{216}$. Using this gives: $p_0 = \frac{4}{3}$, $K = \frac{7}{216}$, $p_1 = \frac{295}{216}$

$$\frac{K^n}{1-K} |p_1 - p_0| < \text{tol} \Rightarrow K^n < \frac{(1-K)\text{tol}}{|p_1 - p_0|}$$

$$n > \frac{\log\left[\frac{(1-K)\text{tol}}{|p_1 - p_0|}\right]}{\log K} = \underline{4.38105}$$

much better estimate!