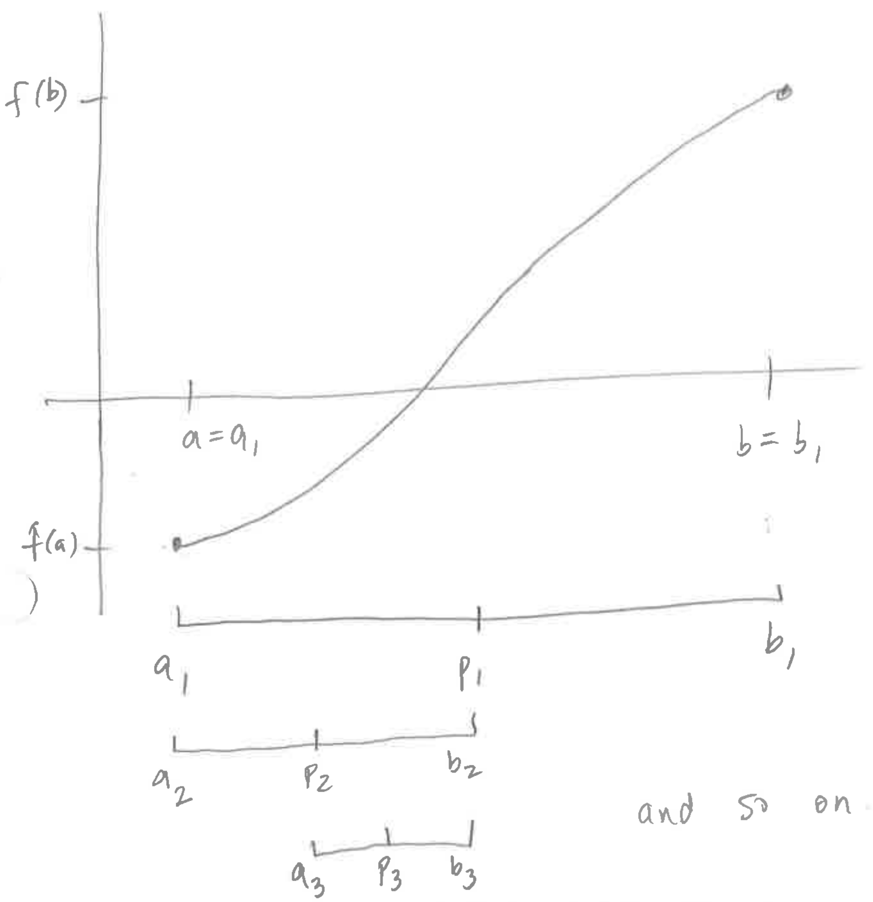
$\boxed{2.1}$ Bisection Method (Binary Search)

Play High/Low.



Algorithm

Input a, b, TOL, max # of iterations N_0

Output approx solution p or message of failure

Step 1 for ($i = 1$ to N_0) do Step 2–4

Step 2 Set $p = \frac{a+b}{2}$

Step 3 if $f(p) = 0$ or $\underbrace{(b-a)/2 < TOL}_{\text{can do others too.}}$ then

output(p);

Stop

Step 4 If $f(a)f(p) > 0$ then set $a = p$

else set $b = p$

Step 5 Output ('Method failed after N_0 iteration!') Stop.

Try on $f(x) = x^2 - 2x + 1 = 0$.

$\boxed{\text{Thm } 2.1}$

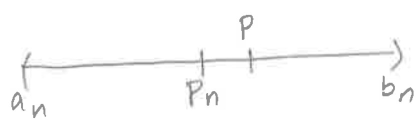
Let $f \in C[a,b]$ and suppose $f(a) \cdot f(b) < 0$.
The Bisection method (Alg. 2.1) generates a sequence $\{p_n\}$ approximating p with the property

$$|p_n - p| \le \frac{b-a}{2^n}, \quad n \ge 1$$

Proof: $\forall\ n \ge 1$, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \qquad \text{(We half the interval each step.)}$$

Note that $p \in (a_n, b_n)$



Since $p_n = \frac{1}{2}(a_n + b_n)$, $\forall\ n \ge 1$, it follows that

$$|p_n - p| \le \frac{1}{2}(b_n - a_n) = \frac{1}{2}\frac{(b-a)}{2^{n-1}} = \frac{b-a}{2^n}$$

Thus, as $n \to \infty$, $p_n \to p$ at the rate of $O(\frac{1}{2^n})$.

Ex: How many iteration of Bisection would be required to get the approx accurate to within 10^{-8}

Suppose $a=0$, $b=1$. Then

$$|p_n - p| \le \frac{b-a}{2^n} < 10^{-8}$$

$$2^{-n} < 10^{-8}$$

$$-n \log 2 < \log 10^{-8}$$

$$n > \frac{8}{\log 2} = 26.57 \Rightarrow 27 \text{ iterations}$$