1.1 Review of Calculus

DEF 1.1: Let f be a function defined on a set X of real numbers; f is said to have the limit L at x_0 if:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that } |f(x) - L| < \varepsilon$$

whenever $x \in X$ and $0 < |x - x_0| < \delta$

DEF 1.2 f is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$

DEF $C(X)$ – set of all functions continuous on X

DEF: Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real or complex numbers. The sequence converges to a number x (called the limit) if

$$\forall \varepsilon > 0, \exists N(\varepsilon) \ni n > N(\varepsilon) \Rightarrow |x_n - x| < \varepsilon$$

Notation: $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$

Thm 1.4 f on X with $x_0 \in X$. The following are equivalent

a. f is continuous at x_0

b. If $\{x_n\}$ is any sequence in X converging to x_0, then

$$\lim_{n \to \infty} f(x_n) = f(x_0)$$

Note: you can also write this as

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f(x_0)$$

DEF 1.5: f def on an open interval containing x_0, f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

When it exists, we define the derivative of f at x_0 to be

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

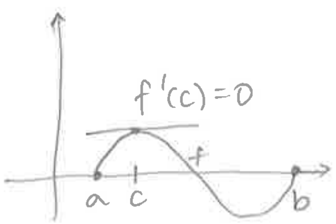
The derivative at x_0 is the slope of the tangent line to the graph at x_0

Thm 1.6: If f is differentiable at x_0, then f is conti at x_0

$C^n[a,b]$ = set of all functions which have continuous n^{th} derivatives on $[a,b]$

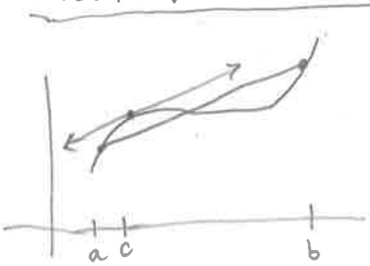
$C^\infty(X)$ = set of all functions which have derivatives of all orders on X.

Rolle's Thm



Suppose $f \in C[a,b]$ and $f \in C'(a,b)$
If $f(a)=f(b)=0$, then $\exists\ c \in (a,b) \ni f'(c)=0$

Mean Value Thm



Suppose $f \in C[a,b]$ and $f \in C'(a,b)$.
Then $\exists\ c \in (a,b) \ni$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\boxed{\text{Extreme Value Thm}}$

suppose $f \in C[a,b]$. Then suppose $c_1, c_2 \in (a,b)$ such that $f(c_1) \le f(x) \le f(c_2)$ $\forall x \in [a,b]$

If $f \in C'(a,b)$, then c_1 & c_2 occur at endpts or where $f'=0$

Riemann Integral The Riemann integral of f on the interval $[a,b]$ is the following limit, provided it exists:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(z_i)\,\Delta x_i \,,$$ where

$$a = x_0 \le x_1 \le x_2 \le \cdots \le x_n = b$$

$$\Delta x_i = x_i - x_{i-1}$$

$$z_i \in [x_{i-1}, x_i]$$

If we make the spacing even, then

$$x_i = a + i \underbrace{\left(\frac{b-a}{n}\right)}_{\Delta x} = a + i\,\Delta x$$

and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\frac{b-a}{n}}_{\Delta x} \sum_{i=1}^{n} f(x_i)$$

$\boxed{\text{Weighted MVT for Integrals}}$ If $f \in C[a,b]$, g is integrable on $[a,b]$, g does not change sign on $[a,b]$,

then $\exists\, c \in (a,b)$ with

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

when $g(x) = 1$, then

$$\int_a^b f(x)\,dx = f(c)\int_a^b dx = f(c)(b-a)$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$ (avg value of a function)

$\boxed{\text{Generalized Rolle's Thm}}$ Let $f \in C[a,b]$ and $f \in C^n(a,b)$

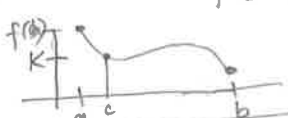
If f vanishes at the $n+1$ distinct pts $x_0, \cdots, x_n$ in $[a,b]$,

then $\exists\, c \in (a,b) \ni f^{(n)}(c) = 0$.

(Note f vanishes means $f(x_0) = f(x_1) = \cdots = f(x_n) = 0$.)

$\boxed{\text{IVT}}$ If $f \in C[a,b]$ and K is any number between $f(a)$ and $f(b)$, then $\exists\, c \in (a,b) \ni f(c) = K$.

Intermediate Value Thm



Ex: Show $f(x) = x^5 - 2x^3 + 3x^2 - 1 = 0$ has a root on $[0,1]$

$f(0) = -1$
$f(1) = 1 - 2 + 3 - 1 = 1$

$\Bigg\}\ \ni 0$ in between

So $f(c) = 0$ for some $c \in [0,1]$

Thm: <u>Taylor's Thm</u>

suppose $f \in C^n[a,b]$, $f^{(n+1)}$ exists on (a,b), $x_0 \in [a,b]$
$\forall\, x \in [a,b]$, $\exists\, \xi(x)$ between x_0 and x with

$$f(x) = \underbrace{P_n(x)}_{\substack{\text{Taylor}\\ \text{polynomial}}} + \underbrace{R_n(x)}_{\substack{\text{Remainder}\\ \text{term}}}$$

where
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{and} \quad R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$

Let $n \to \infty$, Then $P_n(x)$ converges to the Taylor series for $f(x)$.
Let $x_0 = 0$. The Taylor series is then referred to as the Maclaurin Series

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<u>Example</u>

$f(x) = e^x$
Apply Taylor's thm to two terms + error about $x_0 = 0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{P_1(x)} + \frac{1}{2!} f''(\xi)(x-x_0)^2$$

$$= 1 + 1(x-0) + \frac{1}{2} f''(\xi)(x-0)^2$$

$$= 1 + x + \frac{x^2}{2} f''(\xi)$$

$$\boxed{f(x) = 1 + x + \frac{x^2 e^{\xi}}{2}}$$

Estimate the error

$f(x) - P_1(x)$ for $-1 \le x \le 1$

$$E_1(x) = \frac{f''(\xi)}{2}(x-0)^2 = \frac{e^{\xi} x^2}{2}$$

Take abs values

$$|f(x) - P_1(x)| = |E_1(x)| = \left|\frac{e^{\xi} x^2}{2}\right| = \left|\frac{e^{\xi}}{2}\right| |x^2| = \frac{e^{\xi} x^2}{2}$$

Find an upper bound on e^{ξ} (use extreme value thm)

Since $-1 \le x \le 1$ and ξ is between 0 and x, then

$$-1 \le \xi < 1$$

$$e^{-1} \le e^{\xi} < e^{1} \Rightarrow \text{so}$$

If $x \in [-1, 1]$, then $|E_1(x)| = \frac{e^{\xi}}{2} x^2 \le \frac{e}{2} x^2 = c x^2$,

where $\boxed{C = \frac{e}{2}}$

"Big O" notation

$$\Rightarrow e^x = \underbrace{1 + x}_{P_1(x)} + \underbrace{O(x^2)}_{\text{means}}$$

$O(x^2) \le c x^2$ for some const. c.

In general, $f(x) = O(g(x))$ as $x \to 0$ means

$|f(x)| \le c|g(x)|$, whenever x is sufficiently small.

for fixed x,

$$|e^x - P_n(x)| = O\left(\frac{1}{(n+1)!}\right) \quad \text{as } n \to \infty \quad \text{means}$$

$$|e^x - P_n(x)| \le \frac{c}{(n+1)!}, \quad \text{where } c = \max_{\xi \in [-1,1]} f^{(n+1)}(\xi) |1|^{n+1}$$

"Little o" Notation

Taylor Approx to $\cos x$ about $x_0 = 0$
Take degree of poly to be 2

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2}}_{P_2(x)} + \frac{f^{(3)}(\xi)}{3!}(x-x_0)^3$$

$$\cos x = \cos(0) - \sin(0)(x-0) - \frac{\cos(0)(x-0)^2}{2} + \frac{\sin(\xi)}{6}(x-0)^3$$

$$= 1 - \frac{x^2}{2} + \frac{1}{6}\sin(\xi)x^3$$

Since $-1 < \sin(\xi) < 1$, then $\frac{1}{6}\sin(\xi)x^3 = O(x^3)$

$$= 1 - \frac{x^2}{2} + O(x^3)$$

Note: Take 4 terms (instead of 3)

$$\cos x = 1 + \underbrace{0x}_{0} - \frac{1}{2}x^2 + \underbrace{0x^3}_{0} + O(x^4)$$

$$= 1 - \frac{x^2}{2} + O(x^4)$$

So $\cos x - \left(1 - \frac{x^2}{2}\right) = O(x^4)$ ← big

$= o(x^3)$ ← little ← means the LHS goes to zero faster than $c \cdot x^3$