Math 113: Trigonometric Identities
Chapter 6, 7, 8, 9

Many of the trigonometric identities can be derived in succession from the identities:

\[ \sin(-x) = -\sin x, \]  
(1)  
\[ \cos(-x) = \cos x, \]  
(2)  
\[ \sin(x + y) = \sin x \cos y + \sin y \cos x, \]  
(3)  
\[ \cos(x + y) = \cos x \cos y - \sin x \sin y. \]  
(4)

The first and second identities indicate that \( \sin \) and \( \cos \) are odd and even functions, respectively.

Suppose that \( y = -w \), then (3) simplifies to

\[ \sin(x + (-w)) = \sin x \cos(-w) + \sin(-w) \cos x \]  
by (3)  
\[ = \sin x \cos w - \sin w \cos x \]  
by (1) and (2)

Since \( w \) is an arbitrary label, then \( y \) will do as well. Hence,

\[ \sin(x - y) = \sin x \cos y - \sin y \cos x \]  
(5)

Similarly, equation (4) simplifies as

\[ \cos(x - y) = \cos x \cos y + \sin x \sin y \]  
(6)

The Double Angle identities can be derived from equations (3) and (4). Suppose \( x = y \), then (3) simplifies as

\[ \sin(x + x) = \sin x \cos x + \sin x \cos x. \]  
Hence,

\[ \sin(2x) = 2 \sin x \cos x. \]  
(7)

Similarly,

\[ \cos(2x) = \cos^2 x - \sin^2 x. \]  
(8)

The first of the Pythagorean identities can be found by setting \( x = y \) in (6). Hence,

\[ \cos(x - x) = \sin x \sin x + \cos x \cos x. \]  
So,

\[ \sin^2 x + \cos^2 x = 1. \]  
(9)

Dividing both sides of (9) by \( \cos^2 x \) yields

\[ \tan^2 x + 1 = \sec^2 x. \]  
(10)

Dividing both sides of (9) by \( \sin^2 x \) yields

\[ 1 + \cot^2 x = \csc^2 x. \]  
(11)

Equations (8) and (9) can generate the Power Reductions formulas. Using \( \cos^2 x = 1 - \sin^2 x \), (8) can be written as

\[ \cos(2x) = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x. \]
Solving the above equation for $\sin^2 x$ yields

$$\sin^2 x = \frac{1 - \cos(2x)}{2}. \quad (12)$$

Similarly,

$$\cos^2 x = \frac{1 + \cos(2x)}{2}. \quad (13)$$

The product identities can be found using equations (3) through (6). For example, adding (3) and (5) yields

$$\sin(x - y) + \sin(x + y) = \sin x \cos y + \sin y \cos x + \sin x \cos y - \sin x \cos y$$

$$\sin(x - y) + \sin(x + y) = 2 \sin x \cos y.$$ 

Hence,

$$\sin x \cos y = \frac{1}{2} \left[ \sin(x - y) + \sin(x + y) \right]. \quad (14)$$

Similarly,

$$\cos x \cos y = \frac{1}{2} \left[ \cos(x - y) + \cos(x + y) \right] \text{ and} \quad (15)$$

$$\sin x \sin y = \frac{1}{2} \left[ \cos(x - y) - \cos(x + y) \right]. \quad (16)$$