# Math 334

Hypothetical Exam 3, (Chapter 6, 7 in Zill, 5th Ed.)

S. K. Hyde Name:______________________________

*Show all your work to receive credit. All answers must be justified to get full credit.*

These questions are intended to give you some idea of the types of questions which could be asked on your exam. They may not cover all of the topics which will be on your exam (and they may cover more topics than are on your exam). The length of your exam may be shorter than this practice exam (or longer). Working these problems is not a substitute for studying your notes and reading the book.

## Selected Laplace Transforms

The following can be used without proof, unless specifically asked.

| $f(t)$ | $\mathscr{L}\{f(t)\}$ |
|---|---|
| $1$ | $\dfrac{1}{s}$ |
| $t$ | $\dfrac{1}{s^2}$ |
| $t^n$ | $\dfrac{n!}{s^{n+1}},\ n \in \mathbb{Z}^+$ |
| $t^{-1/2}$ | $\sqrt{\dfrac{\pi}{s}}$ |
| $t^{1/2}$ | $\dfrac{\sqrt{\pi}}{2s^{3/2}}$ |
| $t^\alpha$ | $\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \alpha > -1$ |
| $\sin(kt)$ | $\dfrac{k}{s^2+k^2}$ |
| $\cos(kt)$ | $\dfrac{s}{s^2+k^2}$ |
| $\sin^2(kt)$ | $\dfrac{2k^2}{s(s^2+4k^2)}$ |
| $\cos^2(kt)$ | $\dfrac{s^2+2k^2}{s(s^2+4k^2)}$ |
| $e^{at}$ | $\dfrac{1}{s-a}, s > a$ |
| $\sinh(kt)$ | $\dfrac{k}{s^2-k^2}$ |
| $\cosh(kt)$ | $\dfrac{s}{s^2-k^2}$ |

| $f(t)$ | $\mathscr{L}\{f(t)\}$ |
|---|---|
| $\delta(t)$ | $1$ |
| $\delta(t-t_0)$ | $e^{-st_0}$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $f(t-a)\,\mathscr{U}(t-a)$ | $e^{-as}F(s)$ |
| $\mathscr{U}(t-a)$ | $\dfrac{e^{-as}}{s}$ |
| $f^{(n)}(t)$ | $s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$ |
| $t^nf(t)$ | $(-1)^n\dfrac{d^n}{ds^n}F(s)$ |
| $\int_0^t f(\tau)g(t-\tau)d\tau$ | $F(s)G(s)$ |
| $f(t)$ is periodic | $\dfrac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\ dt$ |

## True or False

Circle T or F corresponding to the best answer. (2 pts each)

1. **T F** The function $f(t) = \frac{1}{x-5}$ is of exponential order.
2. **T F** The function $f(t) = \ln t$ is a piecewise continuous function over $[0, \infty)$.
3. **T F** If $f$ is not piecewise continuous on $[0, \infty)$, then $\mathscr{L}\{f(t)\}$ will not exist.

## Show Your Work

Show all work clearly and neatly. No work shown means no credit will be given. Show all steps! No "miracle" steps are allowed. Use correct notation to get full credit. Reserve scratch paper work for scratch paper, which means only include necessary work on the exam. Erase all mistakes neatly. Keep it neat!

Work all the following problems on the **blank COLORED** sheets of paper (not white paper or scratch paper) provided by the testing center personnel. If you need more, go back and get more from the testing center personnel. I will require you to do the following:

1. Start each numbered problem on a **NEW** sheet of paper (Do not start a problem on the back side of a paper!). (5 pts)
2. Make sure the problem numbers are clearly labeled (even after pages are stapled together (e.g. **not covered by a staple**)) (5 pts)
3. Make sure the problems are in **correct numerical/alphabetical order**. (5 pts)

4. (10 pts each). Compute the following Laplace Transforms or Inverse Laplace Transforms.

   (a) $\mathscr{L}\{\cos(kt)\}$

   (b) $\mathscr{L}^{-1}\left\{\dfrac{e^{-5s}}{(s-2)^4}\right\}$

   (c) $\mathscr{L}^{-1}\left\{\dfrac{5s-2}{s^3(s^2+1)}\right\}$

   (d) $\mathscr{L}\{t^2e^{-3t}\sinh t\}$

   (e) $\mathscr{L}\left\{\displaystyle\int_0^t (t-\beta)^3 e^{\beta}\,d\beta\right\}$

   (f) $\mathscr{L}\{f(t)\}$, where $f(t) = \begin{cases} \sin(2t), & 0 < t < \pi \\ \frac{1}{2}, & t = \pi \\ 0, & t > \pi \end{cases}$

   (g) $\mathscr{L}\{t\,\mathscr{U}(t-2)\}$

   (h) $\mathscr{L}\{t^2 * te^t\}$

   (i) $\mathscr{L}\{f(t)\}$, where $f(t)$ is the periodic function in figure 4

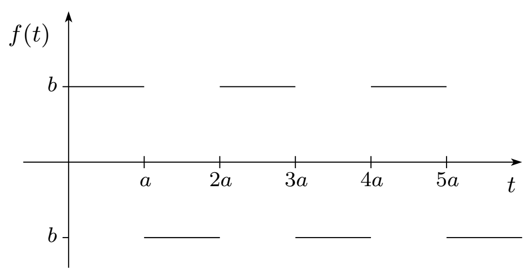


Figure 1:

5. (10 pts) Using the power series method, find two linearly independent power series solutions about the ordinary point $x = 0$ for

$$y'' + x^2y = 0.$$

6. (10 pts) Use Laplace transforms to solve the differential equation

$$y'' + 4y = \delta(t-5)$$

7. (10 pts) Use Laplace transforms to solve the differential equation

$$y'' + 4y = \sin t\,\mathscr{U}(t-2\pi)\,.$$

8. (10 pts) Evaluate: $\displaystyle\int_0^{\infty} t^4e^{3t}\cos(t\pi)\,\delta(t-2)\,dt$

9. (10 pts) Prove the commutative property of the convolution integral ($f * g = g * f$).

10. (10 pts) Solve the differential equation $x^2y'' - 7xy' + 41y = 0$.

11. (10 pts) Solve the differential equation $xy'' - y' = 0$.

12. Given

$$f(t) = \begin{cases} t^2 + 2, & 0 < t < 2 \\ 3t + 2, & 2 < t < 3 \\ 9, & 3 < t \end{cases},$$

answer the following questions:

(a) (10 pts) Express $f(t)$ in terms of Unit Step functions ($\mathscr{U}(t-a)$)

(b) (10 pts) Find the Laplace transform of $f(t)$

13. (10 pts) For $a > 0$, show that, from $\mathscr{L}^{-1}\{F(s)\} = f(t)$, it follows that $\mathscr{L}^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$

14. (10 pts). Using the properties of the Derivatives of Transforms (use $n = 1$ in the formula which yields $f(t) = -\frac{1}{t}\mathscr{L}^{-1}\{\frac{d}{ds}F(s)\}$), find $\mathscr{L}^{-1}\left\{\ln\left(\frac{s-3}{s+1}\right)\right\}$.

15. (10 pts) Use the Laplace transform to solve for $f(t)$ in the integral equation

$$F(t) = 4t - 3\int_0^t F(\beta)\sin(t-\beta)d\beta.$$

16. (10 pts extra credit). Prove the stronger result that if $F(s) = \mathscr{L}\{f(t)\}$ is the Laplace transform of a function that is piecewise continuous and of exponential order, then $\lim_{s\to\infty} sF(s) = c$, where $c$ is a constant.

17. (10 pts extra credit). Prove $\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(S)G(s)$

18. (10 pts extra credit). Suppose $f(t)$ is a piecewise continuous function on $[0, \infty)$ and of exponential order. Prove that if $f(t)$ is periodic with period $T$, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}}\int_0^T e^{-st} f(t)dt$$

19. (10 pts extra credit). Use the Laplace transform to solve the differential equation

$$y'' + 2y' + y = 0$$

subject to the boundary conditions, $y'(0) = 2$, $y(1) = 2$.