Math 221: The Chi-Square Test for Goodness of Fit

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Chapter 22 (Moore, 5th Ed.)

The main objective of this section is to determine whether the distribution of a data set "fits" some claimed distribution. A multinomial experiment is defined similarly to that of a binomial experiment, except that there may be several categories rather than just two. The following notation is used:

O	$=$	represents the observed frequency of an outcome
E	$=$	represents the expected frequency of an outcome
k	$=$	represents the number of different categories or outcomes
n	$=$	represents the total number of trials

The test statistic is $\chi^2 = \sum \frac{(O-E)^2}{E}$, with $k-1$ degrees of freedom.

Example 1:

If you drill a hole in a die and fill it with a lead weight, the die is no longer fair. To test whether or not it is fair, do an experiment: Roll the die 200 times and perform a test of hypothesis. For example, one such outcome was:

Side Showing	1	2	3	4	5	6
Observed Freq	27	31	42	40	28	32

To compute the expected frequency, we note that the expected frequency for category i is $E = np_i$, where p_i is the probability of the i^{th} category. Hence

Side Showing	1	2	3	4	5	6
p_i	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
E	$\frac{200}{6}$	$\frac{200}{6}$	$\frac{200}{6}$	$\frac{200}{6}$	$\frac{200}{6}$	$\frac{200}{6}$

Test the claim that the die is a fair die:

H_0:

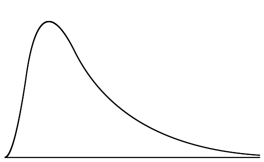
H_a:

$\alpha =$

Test Statistic:

$$\chi^2 = \sum \frac{(O-E)^2}{E} =$$

P**-value**:



Decision: (Circle one:) Reject H_0 or Fail to Reject H_0

Conclusion:

Solution:

H_0: $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$
H_a: At least one of the p_i's is not $\frac{1}{6}$
$\alpha = 0.05$
Test Statistic: $\chi^2 = \frac{(27-200/6)^2}{200/6} + \frac{(31-200/6)^2}{200/6} + \frac{(42-200/6)^2}{200/6} + \frac{(40-200/6)^2}{200/6} + \frac{(28-200/6)^2}{200/6} + \frac{(32-200/6)^2}{200/6} = 5.86$
P**-value**: P-value $= 0.3201$
Decision: Since P-value > 0.05, then Fail to Reject H_0
Conclusion: The data seem to fit the assumed distribution. It appears that there is no reason to believe that the die is not fair.

Note: this procedure is found in the TI-84, but not in the TI-83. However, you can use the TI-83 to get the test statistic and the p-value anyway. First, I'll give directions on using the TI-84.

TI-84

1. Go to the Stat Edit menu `STAT` ⟶ `EDIT` ⟶ `Edit...`
2. Enter the Observed Values into list `L1`. For this example, enter `27, 31, 42, 40, 28, 32`.
3. Enter the Expected Values into list `L2`. For this example, enter `200/6` six times.
4. Select `STAT` ⟶ `EDIT` ⟶ χ^2`GOF-Test`
5. Make sure the degrees of freedom is $k-1$ (in this case $k = 6$, so `df=5`).
6. Select `Calculate`

TI-83

1. Follow steps 1-3 for the TI-84
2. Type the `(L1-L2)`^{2}`/L2` into the main calculator prompt. You can use the following keystrokes

   `(` ⟶ `2ND` ⟶ `L1` ⟶ `-` ⟶ `2ND` ⟶ `L2` ⟶
   `)` ⟶ x^2 ⟶ `/` ⟶ `2ND` ⟶ `L2` ⟶ `ENTER`

3. We now want to sum the answer. So we type:

   `2ND` ⟶ `LIST` ⟶ `MATH` ⟶ `sum(` ⟶ `2ND` ⟶ `ANS` ⟶ `ENTER`

   For this example, the calculator gives 5.86 for the test statistic.

4. To find the p-value, first remember that $k-1$ is the degrees of freedom. The p-value is one minus the area from 0 to the test statistic. To translate that to TI, type `1-`χ^2`cdf(0,Ans,df)`. For this example, the degrees of freedom was 5. Note that `Ans` gets the last answer on the stack (5.86 in this example). The keystrokes to enter the expression above is

   `1` ⟶ `-` ⟶ `2ND` ⟶ `DISTR` ⟶ χ^2`cdf` ⟶ `0` ⟶ `,` ⟶
   `2ND` ⟶ `Ans` ⟶ `,` ⟶ `enter df` ⟶ `)` ⟶ `ENTER`

   The p-value in this case is .3200808017.

Example 2:

According to Benford's law, a variety of different data sets include numbers with leading (first) digits that follow the distributions given in the table below. Data sets with values having leading digits that conform to Benford's law include stock market values, population sizes, numbers appearing on the front page of a newspaper, amounts on tax returns, lengths of rivers, and check amounts.

When working for the Brooklyn District Attorney, investigator Robert Burton used Benford's law to identify fraud by analyzing the leading digits on 784 checks. Test whether the checks follow Benford's law.

Leading digit	1	2	3	4	5	6	7	8	9
p_i	0.301	0.176	0.125	0.097	0.079	0.067	0.058	0.051	0.046
E	235.984	137.984	98.000	76.048	61.936	52.528	45.472	39.984	36.064
O	0	15	0	76	479	183	8	23	0

H_0:

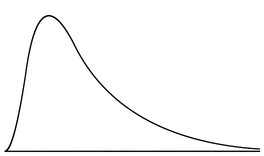
H_a:

$\alpha =$

Test Statistic:

$$\chi^2 = \sum \frac{(O-E)^2}{E} =$$

P**-value**:



Decision: (Circle one:) Reject H_0 or Fail to Reject H_0

Conclusion:

Solution:

H_0: $p_1 = .301,\ p_2 = .176,\ p_3 = .125,\ p_4 = .097,\ p_5 = .079,\ p_6 = .067,\ p_7 = .058,\ p_8 = .051,\ p_9 = .046$
H_a: At least one of the p_i's is not correct
$\alpha = 0.05$
Test Statistic: $\chi^2 = \frac{(0-235.984)^2}{235.984} + \frac{(15-137.984)^2}{137.984} + \frac{(0-98.000)^2}{98.000} + \frac{(76-76.048)^2}{76.048} + \frac{(479-61.936)^2}{61.936} + \frac{(183-52.528)^2}{52.528} + \frac{(8-45.472)^2}{45.472} + \frac{(23-39.984)^2}{39.984} + \frac{(0-36.064)^2}{36.064} = 3650.2514$
P**-value**: P-value $= 0$
Decision: Since P-value < 0.05, Reject H_0
Conclusion: The data do not fit the assumed distribution.

Note: this procedure is found in the TI-84, but not in the TI-83. However, you can use the TI-83 to get the test statistic and the p-value anyway. First, I'll give directions on using the TI-84.

TI-84

1. Go to the Stat Edit menu `STAT` $\longrightarrow$ `EDIT` $\longrightarrow$ `Edit...`
2. Enter the Observed Values into list `L1`. For this example, enter 0, 15, 0, 76, 479, 183, 8, 23, 0.
3. Enter the Expected Values into list `L2`. This is the sum of all the observations times p_i. You can enter the numbers as a multiplication, which makes it easier. For this example, enter `784*.301`, `784*.176`, `784*.125`, `784*.097`, `784*.079`, `784*.067`, `784*.058`, `784*.051`, `784*.046`.

   Another alternative is to enter all the p_i's into `L3`, then move to the top of `L2`, so that `L2` is highlighted, hit enter once, then type `784*L3`. That multiplies 784 by all the entries in `L3` at once, and is easier to see if you've made a mistake entering the p_i's.
4. Select `STAT` $\longrightarrow$ `EDIT` $\longrightarrow$ χ^2`GOF-Test`
5. Make sure the degrees of freedom is $k-1$ (in this case $k = 9$, so $df = 8$).
6. Select `Calculate`

TI-83

1. Follow steps 1-3 for the TI-84
2. Type the `(L1-L2)`^{2}`/L2` into the main calculator prompt. You can use the following keystrokes

   `(` $\longrightarrow$ `2ND` $\longrightarrow$ `L1` $\longrightarrow$ `-` $\longrightarrow$ `2ND` $\longrightarrow$ `L2` $\longrightarrow$
   `)` $\longrightarrow$ x^2 $\longrightarrow$ `/` $\longrightarrow$ `2ND` $\longrightarrow$ `L2` $\longrightarrow$ `ENTER`
3. We now want to sum the answer. So we type:

   `2ND` $\longrightarrow$ `LIST` $\longrightarrow$ `MATH` $\longrightarrow$ `sum(` $\longrightarrow$ `2ND` $\longrightarrow$ `ANS` $\longrightarrow$ `ENTER`

   For this example, the calculator gives 3650.251371 for the test statistic.
4. To find the p-value, first remember that $k-1$ is the degrees of freedom. So we type `1-`χ^2`cdf(0,Ans,df)`. For this example, the degrees of freedom was 8. Note that `Ans` gets the last answer on the stack (3650.251371 in this example). The keystrokes to enter the expression above is

   `1` $\longrightarrow$ `-` $\longrightarrow$ `2ND` $\longrightarrow$ `DISTR` $\longrightarrow$ χ^2`cdf` $\longrightarrow$ `0` $\longrightarrow$ `,` $\longrightarrow$
   `2ND` $\longrightarrow$ `Ans` $\longrightarrow$ `,` $\longrightarrow$ `enter df` $\longrightarrow$ `)` $\longrightarrow$ `ENTER`

   The p-value in this case is 0.