Math 221: The Chi-Square Test for Goodness of Fit
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Chapter 22 (Moore, 5th Ed.)

The main objective of this section is to determine whether the distribution of a data set “fits” some claimed distribution. A multinomial experiment is defined similarly to that of a binomial experiment, except that there may be several categories rather than just two. The following notation is used:

| O     | represents the observed frequency of an outcome |
| E     | represents the expected frequency of an outcome |
| k     | represents the number of different categories or outcomes |
| n     | represents the total number of trials |

The test statistic is \( \chi^2 = \sum \frac{(O-E)^2}{E} \), with \( k - 1 \) degrees of freedom.

**Example 1:**

If you drill a hole in a die and fill it with a lead weight, the die is no longer fair. To test whether or not it is fair, do an experiment: Roll the die 200 times and perform a test of hypothesis. For example, one such outcome was:

<table>
<thead>
<tr>
<th>Side Showing</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed Freq</td>
<td>27</td>
<td>31</td>
<td>42</td>
<td>40</td>
<td>28</td>
<td>32</td>
</tr>
</tbody>
</table>

To compute the expected frequency, we note that the expected frequency for category \( i \) is \( E = np_i \), where \( p_i \) is the probability of the \( i \)th category. Hence

<table>
<thead>
<tr>
<th>Side Showing</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_i )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>( E )</td>
<td>( \frac{200}{6} )</td>
<td>( \frac{200}{6} )</td>
<td>( \frac{200}{6} )</td>
<td>( \frac{200}{6} )</td>
<td>( \frac{200}{6} )</td>
<td>( \frac{200}{6} )</td>
</tr>
</tbody>
</table>

Test the claim that the die is a fair die:

\( H_0: \)

\( H_a: \)

\( \alpha = \)

**Test Statistic:**

\( \chi^2 = \sum \frac{(O-E)^2}{E} = \)

**P-value:**

![Graph](image)

**Decision:** (Circle one) Reject \( H_0 \) or Fail to Reject \( H_0 \)

**Conclusion:**
Solution:

\[ H_0: \ p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6} \]

\[ H_a: \ \text{At least one of the } p_i \text{'s is not } \frac{1}{6} \]

\[ \alpha = 0.05 \]

Test Statistic:

\[ \chi^2 = \frac{(27-200/6)^2}{200/6} + \frac{(31-200/6)^2}{200/6} + \frac{(42-200/6)^2}{200/6} + \frac{(40-200/6)^2}{200/6} + \frac{(28-200/6)^2}{200/6} + \frac{(32-200/6)^2}{200/6} = 5.86 \]

\[ P\text{-value}: \ P\text{-value} = 0.3201 \]

Decision: Since \( P\text{-value} > 0.05 \), then Fail to Reject \( H_0 \)

Conclusion:
The data seem to fit the assumed distribution. It appears that there is no reason to believe that the die is not fair.

Note: this procedure is found in the TI-84, but not in the TI-83. However, you can use the TI-83 to get the test statistic and the \( p \)-value anyway. First, I’ll give directions on using the TI-84.

**TI-84**

1. Go to the Stat Edit menu \( \text{STAT} \rightarrow \text{EDIT} \rightarrow \text{Edit...} \)
2. Enter the Observed Values into list L1. For this example, enter 27, 31, 42, 40, 28, 32.
3. Enter the Expected Values into list L2. For this example, enter 200/6 six times.
4. Select \( \text{STAT} \rightarrow \text{EDIT} \rightarrow \chi^2\text{GOF-Test} \)
5. Make sure the degrees of freedom is \( k - 1 \) (in this case \( k = 6 \), so df=5).
6. Select Calculate

**TI-83**

1. Follow steps 1-3 for the TI-84
2. Type the \((L1-L2)^2/L2\) into the main calculator prompt. You can use the following keystrokes

\[
\left(1 \rightarrow 2ND \rightarrow L1 \rightarrow \frac{2ND \rightarrow L2}{x^2} \rightarrow \right) \rightarrow \text{enter} \]

3. We now want to sum the answer. So we type:

\[
2ND \rightarrow \text{LIST} \rightarrow \text{MATH} \rightarrow \text{sum(} \rightarrow 2ND \rightarrow \text{Ans} \rightarrow \text{enter} \]

For this example, the calculator gives 5.86 for the test statistic.

4. To find the \( p \)-value, first remember that \( k - 1 \) is the degrees of freedom. The \( p \)-value is one minus the area from 0 to the test statistic. To translate that to TI, type \( 1 - \chi^2\text{cdf}(0,\text{Ans},\text{df}) \). For this example, the degrees of freedom was 5. Note that \( \text{Ans} \) gets the last answer on the stack (5.86 in this example). The keystrokes to enter the expression above is

\[
1 \rightarrow - \rightarrow 2ND \rightarrow \text{DISTR} \rightarrow \chi^2\text{cdf} \rightarrow 0 \rightarrow 7 \rightarrow 2ND \rightarrow \text{Ans} \rightarrow \rightarrow \text{enter df} \rightarrow ) \rightarrow \text{enter} \]

The \( p \)-value in this case is .3200808017.
Example 2:

According to Benford’s law, a variety of different data sets include numbers with leading (first) digits that follow the distributions given in the table below. Data sets with values having leading digits that conform to Benford’s law include stock market values, population sizes, numbers appearing on the front page of a newspaper, amounts on tax returns, lengths of rivers, and check amounts.

When working for the Brooklyn District Attorney, investigator Robert Burton used Benford’s law to identify fraud by analyzing the leading digits on 784 checks. Test whether the checks follow Benford’s law.

<table>
<thead>
<tr>
<th>Leading digit</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i$</td>
<td>0.301</td>
<td>0.176</td>
<td>0.125</td>
<td>0.097</td>
<td>0.079</td>
<td>0.067</td>
<td>0.058</td>
<td>0.051</td>
<td>0.046</td>
</tr>
<tr>
<td>$E$</td>
<td>235.984</td>
<td>137.984</td>
<td>98.000</td>
<td>76.048</td>
<td>61.936</td>
<td>52.528</td>
<td>45.472</td>
<td>39.984</td>
<td>36.064</td>
</tr>
<tr>
<td>$O$</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>76</td>
<td>479</td>
<td>183</td>
<td>8</td>
<td>23</td>
<td>0</td>
</tr>
</tbody>
</table>

$H_0$:  

$H_a$:  

$\alpha = \ldots$

Test Statistic:  

$$\chi^2 = \sum \frac{(O - E)^2}{E} = \ldots$$

$P$-value:

Decision: (Circle one:) Reject $H_0$ or Fail to Reject $H_0$

Conclusion:
Solution:

\[ H_0: \quad p_1 = .301, \quad p_2 = .176, \quad p_3 = .125, \quad p_4 = .097, \quad p_5 = .079, \quad p_6 = .067, \quad p_7 = .058, \quad p_8 = .051, \quad p_9 = .046 \]

\[ H_a: \quad \text{At least one of the } p_i \text{'s is not correct} \]

\[ \alpha = 0.05 \]

**Test Statistic:**

\[
\chi^2 = \frac{(0-235.984)^2}{235.984} + \frac{(15-137.984)^2}{137.984} + \frac{(0-98.000)^2}{98.000} + \frac{(76-76.048)^2}{76.048} + \frac{(479-61.936)^2}{61.936} \\
+ \frac{(183-52.528)^2}{52.528} + \frac{(8-45.472)^2}{45.472} + \frac{(23-39.984)^2}{39.984} + \frac{(0-36.064)^2}{36.064} = 3650.2514
\]

**P-value:**

\[ P\text{-value} = 0 \]

**Decision:**

Since \( P\text{-value} < 0.05 \), Reject \( H_0 \)

**Conclusion:**

The data do not fit the assumed distribution.

Note: this procedure is found in the TI-84, but not in the TI-83. However, you can use the TI-83 to get the test statistic and the \( p \)-value anyway. First, I’ll give directions on using the TI-84.

**TI-84**

1. Go to the Stat Edit menu \[ \text{STAT} \rightarrow \text{EDIT} \rightarrow \text{Edit...} \]
2. Enter the Observed Values into list L1. For this example, enter 0, 15, 0, 76, 479, 183, 8, 23, 0.
3. Enter the Expected Values into list L2. This is the sum of all the observations times \( p_i \). You can enter the numbers as a multiplication, which makes it easier. For this example, enter 784*.301, 784*.176, 784*.125, 784*.097, 784*.079, 784*.067, 784*.058, 784*.051, 784*.046.
   
   Another alternative is to enter all the \( p_i \)’s into L3, then move to the top of L2, so that L2 is highlighted, hit enter once, then type 784*L3. That multiplies 784 by all the entries in L3 at once, and is easier to see if you’ve made a mistake entering the \( p_i \)’s.
4. Select \[ \text{STAT} \rightarrow \text{EDIT} \rightarrow \chi^2\text{GOF-Test} \]
5. Make sure the degrees of freedom is \( k - 1 \) (in this case \( k = 9 \), so \( df = 8 \)).
6. Select Calculate

**TI-83**

1. Follow steps 1-3 for the TI-84
2. Type the \( (L1-L2)^2/L2 \) into the main calculator prompt. You can use the following keystrokes

\[
( \rightarrow 2ND \rightarrow L1 \rightarrow - \rightarrow 2ND \rightarrow L2 \rightarrow \\
) \rightarrow x^2 \rightarrow / \rightarrow 2ND \rightarrow L2 \rightarrow ENTER
\]
3. We now want to sum the answer. So we type:

\[
2ND \rightarrow \text{LIST} \rightarrow \text{MATH} \rightarrow \text{sum} \rightarrow 2ND \rightarrow \text{ANS} \rightarrow \text{ENTER}
\]

For this example, the calculator gives 3650.251371 for the test statistic.
4. To find the \( p \)-value, first remember that \( k - 1 \) is the degrees of freedom. So we type \[ 1 \rightarrow \chi^2\text{cdf}(0,\text{Ans},\text{df}) \]. For this example, the degrees of freedom was 8. Note that \( \text{Ans} \) gets the last answer on the stack (3650.251371 in this example). The keystrokes to enter the expression above is

\[
1 \rightarrow - \rightarrow 2ND \rightarrow \text{DISTR} \rightarrow \chi^2\text{cdf} \rightarrow 0 \rightarrow \rightarrow 7 \rightarrow \\
2ND \rightarrow \text{Ans} \rightarrow , \rightarrow \text{enter df} \rightarrow ) \rightarrow \text{ENTER}
\]

The \( p \)-value in this case is 0.