Math 113

The Weierstrass Substitution

The Weierstrass substitution enables any rational function of the regular six trigonometric functions to be integrated using the methods of partial fractions. It uses the substitution of

$$u = \tan\left(\frac{x}{2}\right). \tag{1}$$

The full method are substitutions for the values of dx, $\sin x$, $\cos x$, $\tan x$, $\csc x$, $\sec x$, and $\cot x$. Using the identity $\tan^2\theta + 1 = \sec^2\theta$, the derivative of (1) is

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = \frac{1}{2}\left[1+\tan^2\left(\frac{x}{2}\right)\right]dx = \frac{1}{2}\left(1+u^2\right)dx.$$

It follows that

$$dx = \frac{2\ du}{1+u^2}. \tag{2}$$

To derive the substitutions for $\sin x$ and the other trigonometric substitutions, refer to figure 1 and use the double angle identitities for $\sin x$ and $\cos x$. The double angle identity for $\sin x$ is

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

and for $\cos x$, the double angle identity is

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right).$$

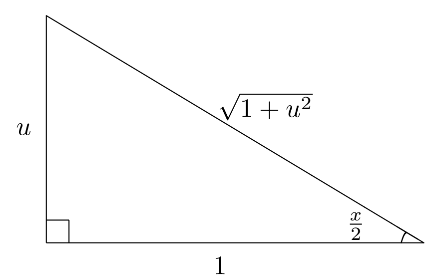

Figure 1: Reference triangle for $u = \tan(\frac{x}{2})$

The substitution for $\sin x$ is

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right) = \frac{2u}{1+u^2} \tag{3}$$

Similarly, for $\cos x$, it is

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 = \frac{1-u^2}{1+u^2} \tag{4}$$

By using (3) and (4), the substitutions for $\tan x$, $\csc x$, $\sec x$, and $\cot x$ is

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{2u}{1+u^2}}{\frac{1-u^2}{1+u^2}} = \frac{2u}{1-u^2} \qquad \csc x = \frac{1}{\sin x} = \frac{1+u^2}{2u}$$

$$\cot x = \frac{1}{\tan x} = \frac{1-u^2}{2u} \qquad \sec x = \frac{1}{\cos x} = \frac{1+u^2}{1-u^2} \tag{5}$$

Note that the resulting equations for all 6 trigonometric functions, along with dx all are simple polynomials in u. Hence, integrals of rational functions of trigonometric functions can be solved using partial fractions. In summary,

$u = \tan\left(\frac{x}{2}\right)$	$\sin x = \frac{2u}{1+u^2}$	$\csc x = \frac{1+u^2}{2u}$	$\tan x = \frac{2u}{1-u^2}$
$dx = \frac{2\ du}{1+u^2}$	$\cos x = \frac{1-u^2}{1+u^2}$	$\sec x = \frac{1+u^2}{1-u^2}$	$\cot x = \frac{1-u^2}{2u}$

Example:

A short example can illustrate the power of the method. The integral of $\sec x$ is known to be

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C,$$

which is found by employing the "trick" of mutiplying the integrand by $\frac{\sec x + \tan x}{\sec x + \tan x}$ and employing the u substitutiton $u = \sec x + \tan x$. A straightforward solution can be found using the Weierstrass method. It follows that

$$\begin{aligned}
\int \sec x \; dx &= \int \frac{1+u^2}{1-u^2}\frac{2}{1+u^2} \; dx \\
&= \int \frac{2}{1-u^2} \; du \\
&= \int \frac{1}{1-u} + \frac{1}{1+u} \; du \\
&= -\ln|1-u| + \ln|1+u| + C \\
&= \ln\left|\frac{1+u}{1-u}\right| + C \\
&= \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + C
\end{aligned}$$

Note that this does not look like $\ln|\sec x + \cos x|$. However, using a bit of trickery (multiply by one) always helps!

$$\begin{aligned}
\int \sec x \; dx &= \ln\left|\frac{1+u}{1-u}\right| + C \\
&= \ln\left|\frac{1+u}{1-u} \cdot \frac{1+u}{1+u}\right| + C \\
&= \ln\left|\frac{(1+u)^2}{1-u^2}\right| + C \\
&= \ln\left|\frac{1+2u+u^2}{1-u^2}\right| + C \\
&= \ln\left|\frac{1+u^2}{1-u^2} + \frac{2u}{1-u^2}\right| + C \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

Hence, the two solutions are identical!