Math 111: Derivation of Trigonometric Identities

Many of the trigonometric identities can be derived in succession from the identities:

\[ \sin(-\theta) = -\sin \theta, \quad (1) \]
\[ \cos(-\theta) = \cos \theta, \quad (2) \]
\[ \sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha, \] and \( (3) \)
\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (4) \]

The first and second identities indicate that \( \sin \) and \( \cos \) are odd and even functions, respectively.

Suppose that \( \beta = -w \), then (3) simplifies to

\[ \sin(\alpha + (-w)) = \sin \alpha \cos(-w) + \sin(-w) \cos \alpha \]
by (3).

\[ = \sin \alpha \cos w - \sin w \cos \alpha \]
by (1) and (2).

Since \( w \) is an arbitrary label, then \( \beta \) will do as well. Hence,

\[ \sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha \quad (5) \]

Similarly, equation (4) simplifies as

\[ \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (6) \]

To find the identity for \( \tan(\alpha + \beta) \), divide \( (3) \) by \( (4) \) as follows:

\[ \tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \sin \beta \cos \alpha}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \quad (7) \]

Divide both the top and bottom of (7) by \( \cos \alpha \cos \beta \) results with

\[ \tan(\alpha + \beta) = \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \beta \cos \alpha}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \quad (8) \]

Because \( \tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta \), then it follows that

\[ \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}. \quad (9) \]

The Double Angle identities can be derived from equations \( (3) \) and \( (4) \). Suppose \( \alpha = \beta = \theta \), then \( (3) \) simplifies as

\[ \sin(\theta + \theta) = \sin \theta \cos \theta + \sin \theta \cos \theta. \]
Hence,

\[ \sin(2\theta) = 2\sin \theta \cos \theta. \quad (10) \]

Similarly,

\[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta, \] and \( (11) \)
\[ \tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta}. \quad (12) \]
The first of the Pythagorean identities can be found by setting $\alpha = \beta = \theta$ in (6). Hence,

$$\cos(\theta - \theta) = \sin \theta \sin \theta + \cos \theta \cos \theta.$$ 

So,

$$\sin^2 \theta + \cos^2 \theta = 1. \quad (13)$$

Dividing both sides of $\text{(13)}$ by $\cos^2 \theta$ yields

$$\tan^2 \theta + 1 = \sec^2 \theta. \quad (14)$$

Dividing both sides of $\text{(13)}$ by $\sin^2 \theta$ yields

$$1 + \cot^2 \theta = \csc^2 \theta. \quad (15)$$

Equations (11) and (13) can generate the Power Reducing identities. Using $\cos^2 \theta = 1 - \sin^2 \theta$, (11) can be written as

$$\cos(2\theta) = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2\sin^2 \theta.$$

Solving the above equation for $\sin^2 \theta$ yields

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}. \quad (16)$$

Similarly,

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}, \quad \text{and} \quad (17)$$

$$\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}. \quad (18)$$

The product identities can be found using equations (3) through (6). For example, adding (3) and (5) yields

$$\sin(\alpha - \beta) + \sin(\alpha + \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha + \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

$$\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2 \sin \alpha \cos \beta.$$

Hence,

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]. \quad (19)$$

Similarly,

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)], \quad (20)$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)], \quad \text{and} \quad (21)$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]. \quad (22)$$
Substituting \( \alpha = \frac{u+v}{2} \) and \( \beta = \frac{u-v}{2} \) into (19) yields:

\[
\frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)] = \sin \alpha \cos \beta
\]

\[\implies \quad \sin(\alpha - \beta) + \sin(\alpha + \beta) = 2 \sin \alpha \cos \beta \]

\[\implies \quad \sin \left( \frac{u+v}{2} - \frac{u-v}{2} \right) + \sin \left( \frac{u+v}{2} + \frac{u-v}{2} \right) = 2 \sin \left( \frac{u+v}{2} \right) \cos \left( \frac{u+v}{2} \right) \]

\[\implies \quad \sin u + \sin v = 2 \sin \left( \frac{u+v}{2} \right) \cos \left( \frac{u+v}{2} \right) \]

Since \( u \) and \( v \) are arbitrary labels, then \( \alpha \) and \( \beta \) will do just as well. Hence,

\[
\sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha + \beta}{2} \right) \quad \text{(23)}
\]

Similarly, replacing \( \alpha \) by \( \frac{\alpha + \beta}{2} \) and \( \beta \) by \( \frac{\alpha - \beta}{2} \) into (20), (21), and (22) yields

\[
\sin \alpha - \sin \beta = 2 \cos \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right) \quad \text{(24)}
\]

\[
\cos \alpha + \cos \beta = 2 \cos \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) \quad \text{(25)}
\]

\[
\cos \alpha - \cos \beta = -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right) \quad \text{(26)}
\]